Digital Signal Processing

Fourier Transform Properties

Properties of Fourier Transform

Linearity property

If $g_1(t) \Leftrightarrow G_1(\omega)$ **and** $g_2(t) \Leftrightarrow G_2(\omega)$

then $a_1 g_1(t) + a_2 g_2(t) \Leftrightarrow a_1 G_1(\omega) + a_2 G_2(\omega)$

where a_1 **and** a_2 **are constants**

This property is proved easily by linearity property of integrals used in defining Fourier transform

Properties of Fourier Transform

Symmetry property

If $g(t) \Leftrightarrow G(\omega)$, then $G(t) \Leftrightarrow 2\pi g(-\omega)$

Proof

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega$$

$$2\pi g(-t) = \int_{-\infty}^{\infty} G(\omega)e^{-j\omega t}\,d\omega$$

we can interchange the variable t and ω, i.e. let $t \to \omega$, $\omega \to t$, then

$$2\pi g(-\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t}\,dt$$

$$\therefore G(t) \Leftrightarrow 2\pi g(-\omega)$$

Properties of Fourier Transform

Time scaling property

$$g(at) \Leftrightarrow \frac{1}{|a|} G(\frac{\omega}{a})$$

Proof

$$F[g(at)] = \int_{-\infty}^{\infty} g(at) e^{-j\omega t} dt$$

let x = at, then dt = dx/a,

case 1: when a > 0,

$$F[g(at)] = \frac{1}{a} \int_{-\infty}^{\infty} g(x) e^{-j\omega x/a} dx = \frac{1}{a} G(\frac{\omega}{a})$$

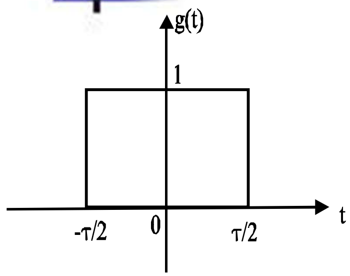
case 2: when a < 0, then t $\rightarrow \infty$ leads to x $\rightarrow - \infty$,

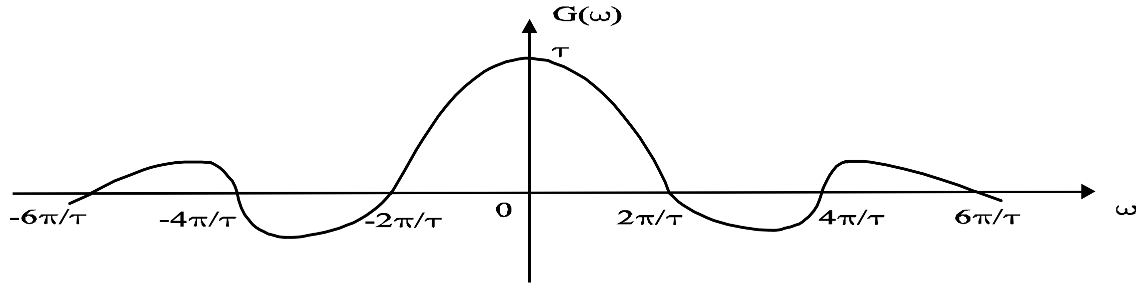
$$F[g(at)] = \frac{1}{a} \int_{\infty}^{-\infty} g(x) e^{-j\omega x/a} dx = -\frac{1}{a} \int_{-\infty}^{\infty} g(x) e^{-j\omega x/a} dx = -\frac{1}{a} G(\frac{\omega}{a})$$

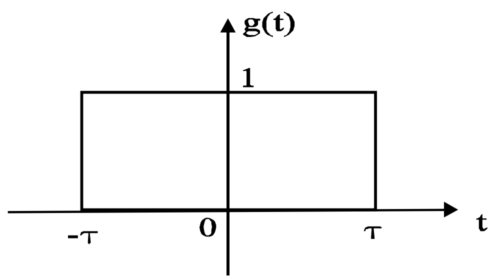
Combined, the two cases are expressed as,

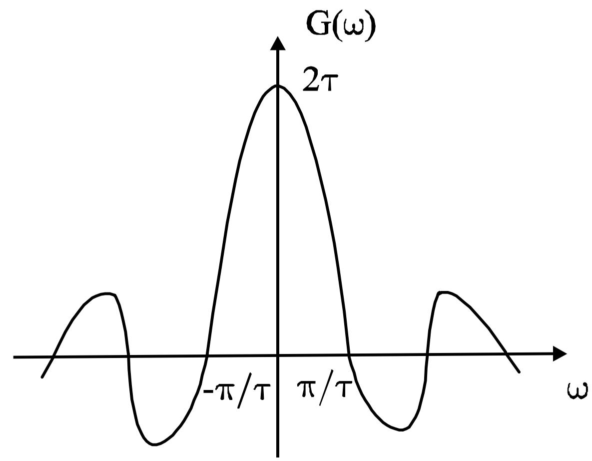
$$g(at) \Leftrightarrow \frac{1}{|a|} G(\frac{\omega}{a})$$

Properties of Fourier Transform









Important Observation:

Time domain ***compression*** *of a signal results in spectral* ***expansion***

Time domain ***expansion*** *of a signal results in spectral* ***compression***

Properties of Fourier Transform

Time shifting property

$$g(t-t_0) \Leftrightarrow G(\omega)e^{-j\omega t_0}$$

Proof

$$F[g(t-t_0)] = \int_{-\infty}^{\infty} g(t-t_0)e^{-j\omega t}dt$$

put t – t_0 = x, so that dt = dx, then

$$F[g(t-t_0)] = \int_{-\infty}^{\infty} g(x)e^{-j\omega(x+t_0)}dx = e^{-j\omega t_0}\int_{-\infty}^{\infty} g(x)e^{-j\omega x}dx = G(\omega)e^{-j\omega t_0}$$

Frequency shifting property

$$g(t)e^{j\omega_0 t} \Leftrightarrow G(\omega-\omega_0)$$

Proof

$$F[g(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}e^{j\omega_0 t}dt = \int_{-\infty}^{\infty} g(t)e^{-j(\omega-\omega_0)t}dt = G(\omega-\omega_0)$$

Properties of Fourier Transform

Significance

- ***Multiplication of a function g(t) by exp(jω_0t) is equivalent to shifting its Fourier transform in the positive direction by an amount ω_0.*** **-- *Frequency translation theorem*.**

- ***Translation of a spectrum*** **helps in achieving *modulation*, which is performed by multiplying the known signal g(t) by a sinusoidal signal.**

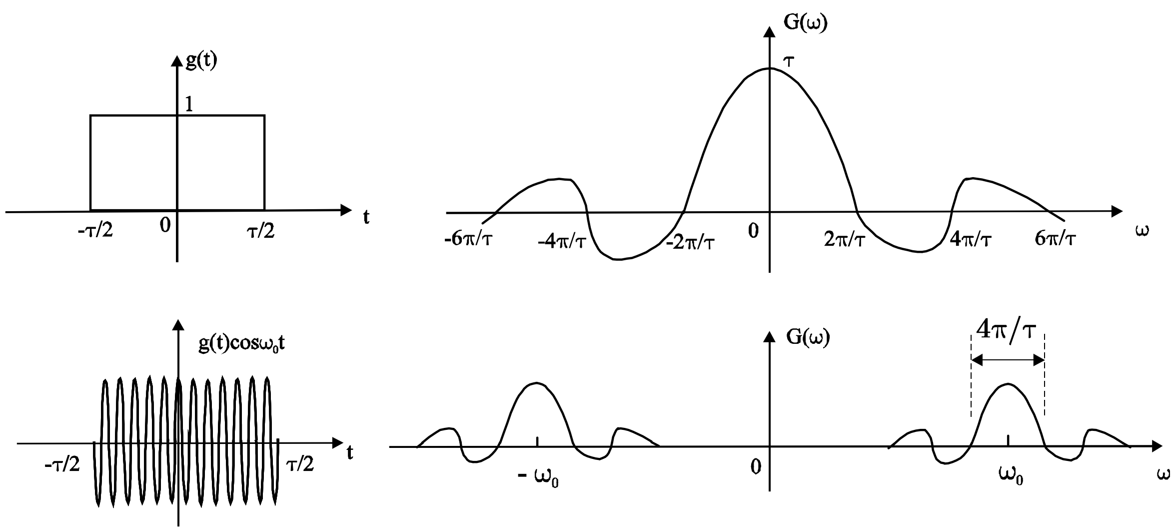
$$g(t)\cos\omega_0 t = \frac{1}{2}[g(t)e^{j\omega_0 t} + g(t)e^{-j\omega_0 t}]$$

Therefore,

$$g(t)\cos\omega_0 t \Leftrightarrow \frac{1}{2}[G(\omega-\omega_0) + G(\omega+\omega_0)]$$

Modulation Theorem

- The multiplication of a time function with a sinusoidal function translates the whole spectrum $G(\omega)$ to $\pm\omega_0$.
- $\exp(j\omega_0 t)$ can also provide frequency translation, but it is not a real signal. Hence, sinusoidal function is used in practical modulation system.



Properties of Fourier Transform

Convolution

Suppose that $g_1(t) \Leftrightarrow G_1(\omega)$ and $g_2(t) \Leftrightarrow G_2(\omega)$, then,

what is the waveform of g(t) whose Fourier transform is the product of $G_1(\omega)$ and $G_2(\omega)$?

This question arises frequently in spectral analysis, and is answered by the *convolution theorem*.

The convolution of two time function $g_1(t)$ and $g_2(t)$, is defined by the following integral

$$g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau$$

Convolution Theorem

Time convolution theorem

If $g_1(t) \Leftrightarrow G_1(\omega)$ and $g_2(t) \Leftrightarrow G_2(\omega)$

Then $g_1(t) * g_2(t) \Leftrightarrow G_1(\omega)G_2(\omega)$

$$F[g_1(t) * g_2(t)] = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} g_1(\tau)[\int_{-\infty}^{\infty} g_2(t-\tau)e^{-j\omega(t-\tau)}dt]e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} g_1(\tau)G_2(\omega)e^{-j\omega\tau}d\tau$$

$$= G_1(\omega)G_2(\omega)$$

Frequency convolution theorem

If $g_1(t) \Leftrightarrow G_1(\omega)$ and $g_2(t) \Leftrightarrow G_2(\omega)$

Then

$$g_1(t)g_2(t) \Leftrightarrow \frac{1}{2\pi} G_1(\omega) * G_2(\omega)$$

The proof is similar to time convolution theorem.

Convolution Theorem: Applications

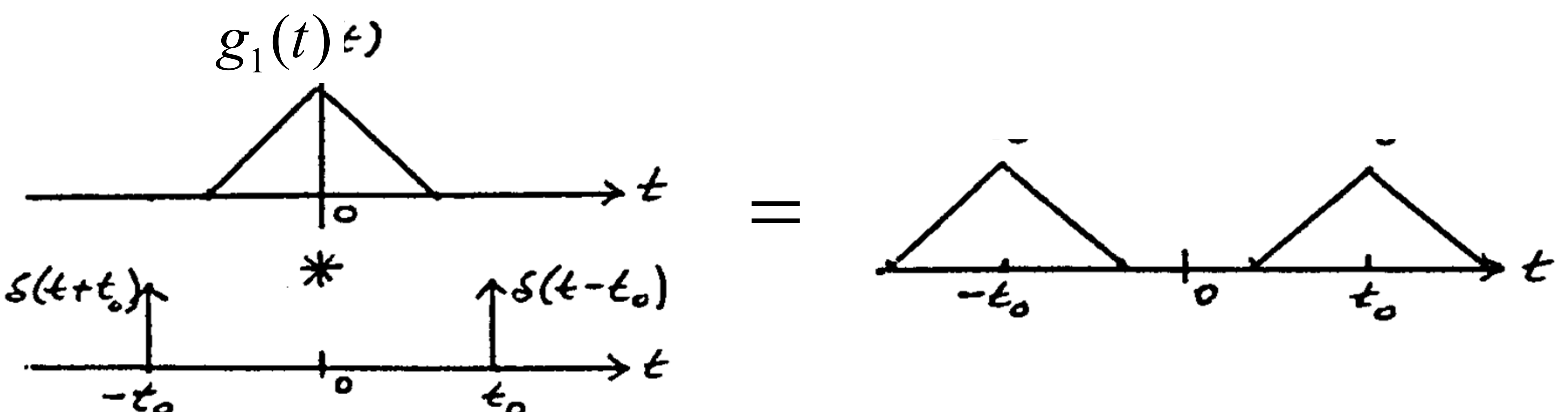
$$g_1(t) * g_2(t) \Leftrightarrow G_1(\omega)G_2(\omega)$$

If we let $g_2(t) = \delta(t - t_0)$, then $g_1(t) * \delta(t - t_0) \Leftrightarrow G_1(\omega)e^{-j\omega t_0}$

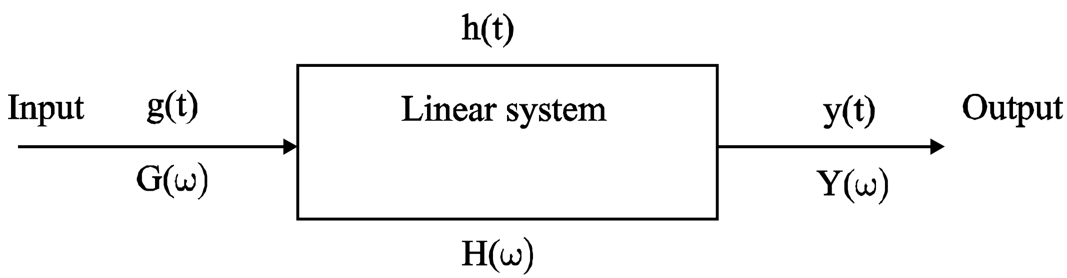
But

$$G_1(\omega)e^{-j\omega t_0} \Leftrightarrow g_1(t - t_0) \qquad \text{(time shifting property)}$$

Therefore, convolving with a delta function shifted in time by t_0 **corresponds to a shift of the original signal by** t_0



Signal transmission through a linear system



Block diagram of a system

$$y(t) = g(t) * h(t)$$

when $g(t) \Leftrightarrow G(\omega)$, $h(t) \Leftrightarrow H(\omega)$, $y(t) \Leftrightarrow Y(\omega)$, $h(t)$ is the impulse response, i.e. if the input is $\delta(t)$, then $y(t) = h(t)$.

By convolution theorem

$$Y(\omega) = G(\omega)H(\omega)$$

where $H(\omega)$ is the system transfer function.

Signal Analysis

Signal power

- **Signal-to-noise ratio (S/N) is an important parameter used to evaluate the system performance.**
- **Noise, being random in nature, cannot be expressed as a time function, like deterministic waveform. It is represented by power.**

Hence, to evaluate the S/N, it is necessary to evolve a method for calculating the signal power.

For a general time domain signal g(t), its *average power* is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \, dt$$

Signal Analysis

For a periodic signal, each period contains a replica of the function, and the limiting operation can be omitted as long as T is taken as the period.

For a real signal

$$P = \overline{g^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)dt$$

Example

Find the power of a sinusoidal signal $\cos\omega_0 t$.

Solution

$$P = \overline{\cos^2(\omega_0 t)} = \frac{1}{T} \int_{-T/2}^{T/2} \frac{1+\cos 2\omega_0 t}{2} dt = \frac{1}{2T}(t + \frac{\sin 2\omega_0 t}{2\omega_0})\Big|_{-T/2}^{T/2} = \frac{1}{2}$$

Is it also possible to determine the signal power in frequency domain?

Frequency domain representation for signals of arbitrary waveshape

When dealing with *deterministic* signals, knowledge of the spectrum implies knowledge of the time domain signal.

For an arbitrary (*random*) signal, Fourier analysis cannot be used because g(t) is not known *analytically*.

For such an undeterministic signal (which include information signals and noise waveforms), the power spectrum $S_g(\omega)$ (or power spectral density) concept is used.

The power spectrum describes the distribution of power versus frequency.

The average signal power is then given by

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_g(\omega)d\omega = \frac{1}{\pi}\int_{0}^{\infty} S_g(\omega)d\omega$$

where $S_g(\omega) > 0$ for all ω.

Another way to evaluate the signal power!

Signal Analysis

Correlation

Correlation **measure of similarity between one waveform, and time delayed version of the other waveform.**

The autocorrelation function is a special case of convolution, and it measures the similarity of a function with its delayed replica, and is given by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) g^{*}(t+\tau) dt$$

Signal Analysis

Important properties of autocorrelation

(1) the *autocorrelation for $\tau = 0$* is *average power of the signal*

$$R(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g^*(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = P$$

The third way to evaluate signal power!

(2) power spectral density $S_g(\omega)$ and autocorrelation function of a power signal are Fourier transform pair

$$R(\tau) \Leftrightarrow S_g(\omega)$$

Exercise Problems (Signal Analysis)

1. Evaluate the integrals

$$\int_{-\infty}^{\infty} e^{\cos t}\delta(t-\pi)dt \qquad \int_{1}^{\infty} e^{-2x}\delta(x)dx \qquad \int_{-\infty}^{\infty} e^{-t}\delta(t+3)dt$$

$$\int_{-\infty}^{\infty} \delta(2t-4)(2t^2+t-8)dt \qquad \int_{-\infty}^{\infty} \cos(9t)\delta(t-2)dt$$

2. Simplify the following expressions:

(a) $[\sin t/(t+2)]\,\delta(t)$; (b) $[1/(j\omega+2)]\,\delta(\omega+3)$;

(c) $[\sin(k\omega)/\omega]\,\delta(\omega)$;

3. Calculate the (a) average value, (b) ac power, and (c) average power of the periodic waveform $v(t) = 1 + \cos\omega_0 t$.

Exercise Problems (Signal Analysis)

4. Prove that $$\delta(at) = \frac{1}{|a|}\delta(t)$$

5. If $g(t) \Leftrightarrow G(\omega)$, then show that $g^*(t) \Leftrightarrow G^*(-\omega)$.

6. Find the Fourier transform of the signal $f(t) = [A + f_m(t)]\cos\omega_c t$ if $f_m(t)$ has a spectrum $F_m(\omega)$.

7. If $f(t)$ has a spectrum $F(\omega)$, find the Fourier transform of the following functions: (a) $f(t/2 - 5)$; (b) $f(3 - 3t)$; (c) $f(2 + 5t)$;

8. Determine the average power of the following signals:
(a) $A\cos\omega_0 t + B\sin\omega_0 t$; (b) $(A + \sin\omega_0 t)\cos\omega_0 t$;

Math. Table

Properties of Fourier Transform

Linearity: $a_1g_1(t) + a_2g_2(t) \Leftrightarrow a_1G_1(\omega) + a_2G_2(\omega)$

Symmetry: If $g(t) \Leftrightarrow G(\omega)$, then $G(t) \Leftrightarrow 2\pi g(-\omega)$

Time scaling: $g(at) \Leftrightarrow \frac{1}{|a|}G(\frac{\omega}{a})$

Time shifting: $g(t-t_0) \Leftrightarrow G(\omega)e^{-j\omega t_0}$

Frequency shifting: $g(t)e^{j\omega_0 t} \Leftrightarrow G(\omega-\omega_0)$

Modulation theorem: $g(t)\cos\omega_0 t \Leftrightarrow \frac{1}{2}[G(\omega-\omega_0)+G(\omega+\omega_0)]$

Time convolution: $g_1(t) * g_2(t) \Leftrightarrow G_1(\omega)G_2(\omega)$

Frequency convolution: $g_1(t)g_2(t) \Leftrightarrow \frac{1}{2\pi}G_1(\omega)*G_2(\omega)$

Conjugate functions: $g^*(t) \Leftrightarrow G^*(-\omega)$

Time differentiation: $\frac{d}{dt}g(t) \Leftrightarrow j\omega G(\omega)$

Time integration: $\int_{-\infty}^{t} g(\tau)d\tau \Leftrightarrow \frac{1}{j\omega}G(\omega)+\pi G(0)\delta(\omega)$