Digital Signal Processing

Z-Transform & ROC

Introduction to Signals and Systems

Z- Transform

The *z*-Transform is a special case of the Laplace transform and results from applying the Laplace transform to a discrete-time signal.

The *z*-Transform behaves much like the Laplace transform and can be applied to difference equations to produce frequency and time domain responses.

Definition:

The z-transform of a discrete time signal is defined as the power series

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Where z is a complex variable. For convenience, the z-transform of a signal x[n] is denoted by

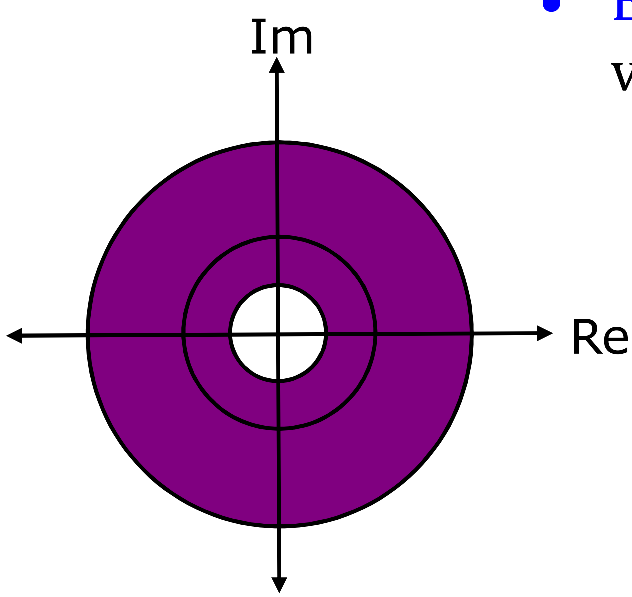
X(z) = Z{x[n]}

Since the z-transform is an infinite series, it exists only for those values of z for which this series converges. The Region of Convergence (ROC) of X(z) is the set of all values of z for which this series converges.

We illustrate the concepts by some simple examples.

Region of Convergence (ROC)

- ROC: The set of values of z for which the z-transform converges
- The region of convergence is made of circles



- Example: z-transform converges for values of $0.5<r<2$

  ROC is shown on the left

  In this example the ROC includes the unit circle, so DTFT exists

Example 1: Determine the z-transform of the following signals:

(a) $x[n] = [1, 2, 5, 7, 0, 1]$

Solution: $X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5,}$

ROC: entire z plane except $z = 0$

(b) $y[n] = [1, 2, 5, \underset{\uparrow}{7}, 0, 1]$

Solution: $Y(z) = z^{2} + 2z + 5 + 7z^{-1} + z^{-3}$

ROC: entire z-plane except $z = 0$ and $z = \infty$.

(c) $z[n] = [0, 0, 1, 2, 5, 7, 0, 1]$

Solution: $z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7,}$ ROC: all z except z=0

(d) $p[n] = \delta[n]$

Solution: $P(z) = 1$, ROC: entire z-plane.

(e) $q[n] = \delta[n - k]$, $k > 0$

Solution: $Q(z) = z^{-k}$, entire z-plane except $z=0$.

(f) $r[n] = \delta[n+k]$, $k > 0$

Solution: $R(z) = z^{k}$,

ROC: entire z-plane except $z = \infty$.

Example 2: Determine the z-transform of $x[n] = (1/2)^n u[n]$

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= 1 + \frac{1}{2} z^{-1} + \left(\frac{1}{2} z\right)^{-2} + \ldots\ldots$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}}$$

ROC: $|1/2\ z^{-1}| < 1$, or equivalently $|z| > 1/2$

7

Example 3: Determine the z-transform of the signal $x[n] = a^n u[n]$

Solution:

Right-Sided Exponential Sequence

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

- For Convergence we require

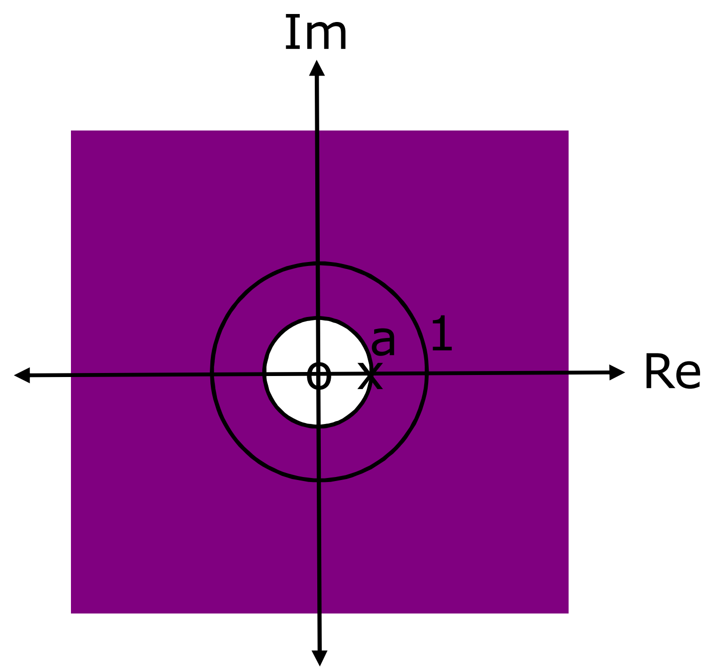
$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$

- Hence the ROC is defined as

$$\left|az^{-1}\right|^n < 1 \Rightarrow |z| > |a|$$

- Inside the ROC series converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$



- Region outside the circle of radius a is the ROC
- Right-sided sequence ROCs extend outside a circle

Example 4: Left-Sided Exponential Sequence

$$x[n] = -a^n u[-n-1]$$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} \left(a^{-1} z\right)^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n$$

$$ROC:$$

$$\sum_{n=0}^{\infty} \left|a^{-1} z\right|^n < \infty \Rightarrow \left|a^{-1} z\right| < 1 \Rightarrow |z| < |a|$$

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

Example 5: Two-Sided Exponential Sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{\left(-\frac{1}{3}z^{-1}\right)^0 - \left(-\frac{1}{3}z^{-1}\right)^\infty}{1+\frac{1}{3}z^{-1}} = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$\sum_{n=-\infty}^{-1}\left(\frac{1}{2}z^{-1}\right)^n = \frac{\left(\frac{1}{2}z^{-1}\right)^{-\infty} - \left(\frac{1}{2}z^{-1}\right)^0}{1-\frac{1}{2}z^{-1}} = \frac{-1}{1-\frac{1}{2}z^{-1}}$$

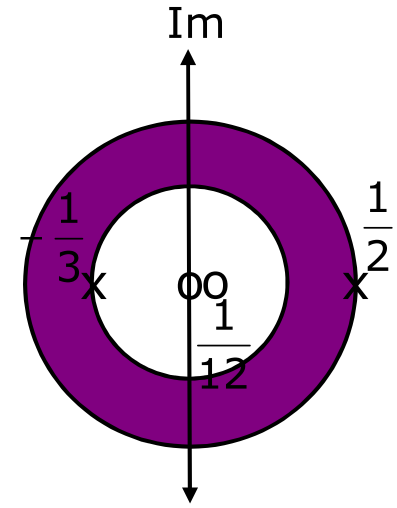
$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$\text{ROC}: \left|-\frac{1}{3}z^{-1}\right| < 1$$

$$\frac{1}{3} < |z|$$

$$\text{ROC}: \left|\frac{1}{2}z^{-1}\right| > 1$$

$$\frac{1}{2} > |z|$$



10

Some common Z-transform pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Some common Z-transform pairs

SEQUENCE	TRANSFORM	ROC
$\delta[n]$	1	ALL z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 (if m > 0) or ∞ (if m < 0)

Some common Z-transform pairs

$$a^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \qquad ROC: \quad |z| > |a|$$

$$a^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \qquad ROC: \quad |z| < |a|$$

$$na^n u[n] \overset{Z}{\longleftrightarrow} \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \qquad ROC: \quad |z| > |a|$$

$$-na^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \qquad ROC: \quad |z| < |a|$$

$$[\cos \omega_0 n] u[n] \overset{Z}{\longleftrightarrow} \frac{1-[\cos \omega_0] z^{-1}}{1-[2\cos \omega_0] z^{-1} + z^{-2}} \qquad ROC: \quad |z| > 1$$