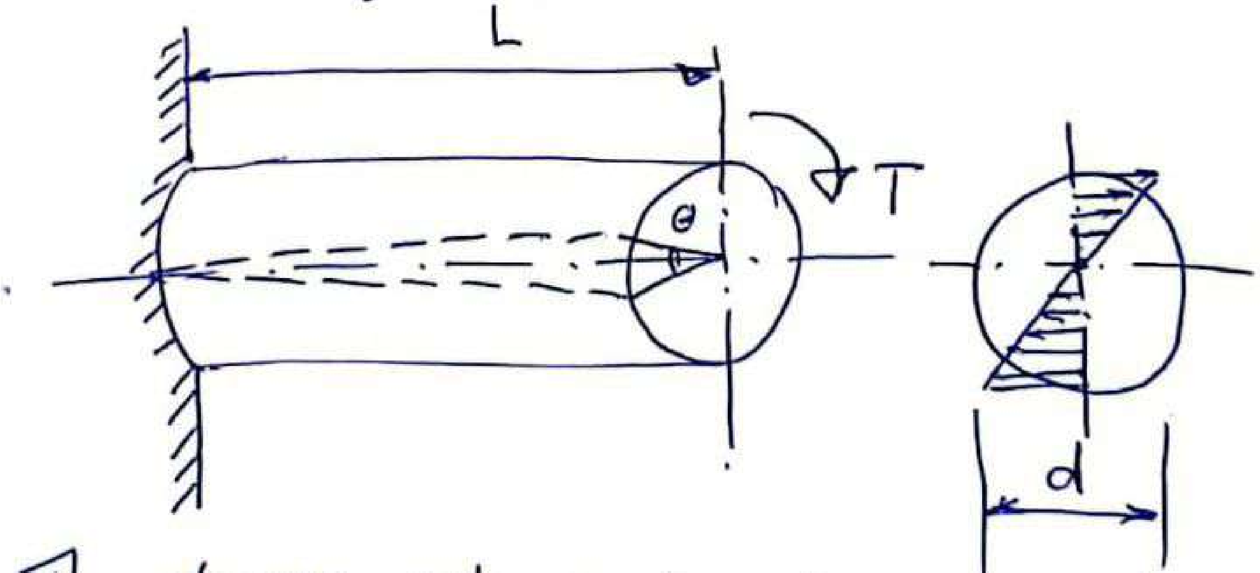
Torsion

When the member is subjected to the action of two equal and opposite couples acting in parallel planes (or torque or twissting moment) then the member is said to be subjected to Torsion.

Consider a shaft fixed at one end and subjected to Torque (T) at the other end as shown:-



Note The stress set up by torsion is know as shear stress, It is zero at the centroidal axis and maximum at the outer surface.

The maximum torsional shear stress at the outer surface of the shaft may obtained from the following equation :-

$$\frac{\tau}{r} = \frac{T}{J} = \frac{G.\theta}{l} \Rightarrow \theta = \frac{T.L}{G\,J}$$

Where,

τ - Torsional shear stress

r - Radius

J - Second moment of area of the section about its polar axis

T - Torque or twisting moment

G - Modulus of rigidity

l - Length

θ - Angle of twist in radians on a length

① $\frac{T}{J} = \frac{\tau}{r} \Rightarrow T = \tau \times \frac{J}{r}$

For a solid of diameter (d), the polar moment of inertia

$$J = I_{xx} + I_{yy} = \frac{\pi d^4}{64} + \frac{\pi d^4}{64} = \frac{\pi d^4}{32}$$

$$\Rightarrow T = \tau \cdot \frac{\pi \cancel{d^4}^{\,3}}{\cancel{32}_{16}} * \frac{\cancel{2}^{\,1}}{\cancel{d}} = \frac{\pi}{16} \tau d^3$$

In case of hollow shaft with external diameter (d_o) and internal diameter (d_i)

The polar moment of inertia

$$J = \frac{\pi}{32}\left(d_o^4 - d_i^4\right) \quad \text{and} \quad r = \frac{d_o}{2}$$

$$\Rightarrow T = \tau \cdot \frac{\pi}{32}\left[d_o^4 - d_i^4\right] * \frac{2}{d_o}$$

$$T = \frac{\pi}{16} \tau \left[\frac{d_o^4 - d_i^4}{d_o}\right]$$

$$T = \frac{\pi}{16} \tau d_o^3 \left(1 - k^4\right) \qquad k = \frac{d_i}{d_o}$$

(2) $$\frac{T}{J} = \frac{G \cdot \theta}{L} \qquad \Rightarrow T = \frac{G \cdot \theta}{L} J$$

$$\text{or} \Rightarrow \theta = \frac{TL}{GJ}$$

<u>Note</u> The similarity of this equation and equation of linear deformation $(\delta) = \dfrac{PL}{AE}$

In many particular application, shaft are use to transmit power, From dynamics, It is known that the power transmitted by a constant torque (T) rotating at a constant angular speed is given by:-

$$Power = T \cdot \omega$$

where ω - angular speed rad/sec.

$$\omega = 2\pi N$$

$$\Rightarrow P = 2\pi N T$$

$$\Rightarrow T = \frac{P}{2\pi N}$$

Power - measured in watts (1W = 1 N.m)

N = Shaft rotational speed (r.p.m)

Ex-1 - A solid shaft transmits 20kW at (2 r/s) Determine the diameter of the Shaft if the shearing stress is not exceed 40 MN/m² and angle of twist is limited to 6° in length of 3m, G = 83 GN/m².

$$P = 2\pi N T$$

$$T = \frac{P}{2\pi N} = \frac{20\times10^{3}}{2\pi(2)} = 1590 \ N.m$$

$$\frac{\tau}{r} = \frac{T}{J} \qquad r = \frac{d}{2}, \ J = \frac{\pi d^4}{32}$$

$$\Rightarrow T = \frac{\pi}{16}\tau d^3$$

$$1590 = \frac{\pi}{16}(40\times10^{6})\, d^3$$

$$\Rightarrow d^3 = 202\times10^{-6} \ m^3 = 202\times10^{3} \ mm^3$$

$$\Rightarrow d = 58.7 \ mm$$

To detemine the (d) necessary to satisfy the requirement of rigidity :-

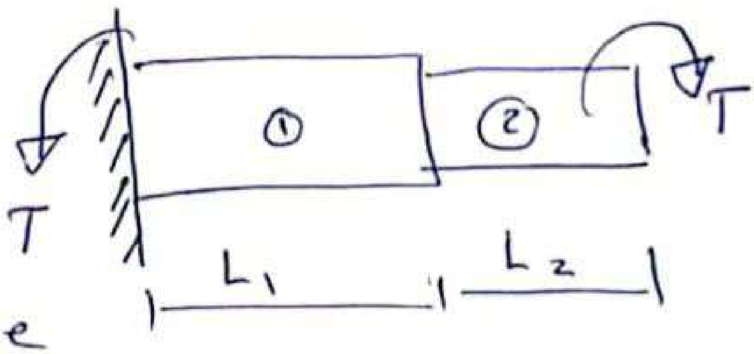
$$\theta = \frac{T.L}{G J} \Rightarrow 6^{\circ} * \frac{\pi}{180} = \frac{1590(3)}{83\times10^{9}\ \frac{\pi d^4}{32}}$$

$$d^4 = 5.59\times10^{-6} \ m^4 = 5.59\times10^{6} \ mm^4 \Rightarrow d = 48.6 \ m$$

Shafts In series and parallel :-

when two shafts of different diameters are connected togther to form one shaft, it is then known as composite shaft.

① if the driving torque is applied at one end and the resisting torque at the other end, then the shafts are said to connected in series as shown:-



In such case each shaft transmits the same Torque and the total angle of twist is equal to the sum of the angle of twists of the two shaft

$$\theta = \theta_1 + \theta_2 = \frac{TL_1}{G_1 J_1} + \frac{TL_2}{G_2 J_2}$$

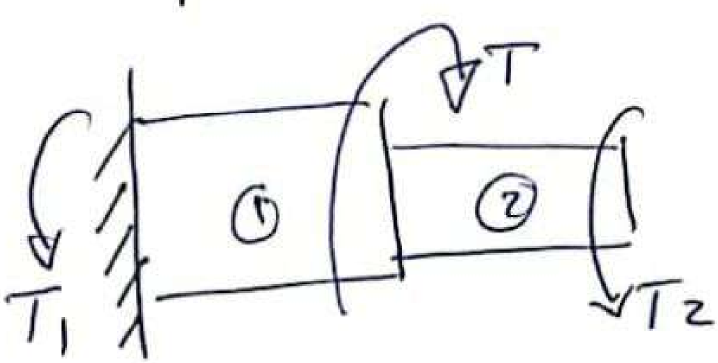
if the shaft are made of the same material

$$G_1 = G_2$$

$$\Rightarrow \theta = \frac{TL_1}{G J_1} + \frac{TL_2}{G J_2} = \frac{T}{G}\left[\frac{L_1}{J_1} + \frac{L_2}{J_2}\right]$$

② When the driving torque (T) is applied at the junction of the two shafts and the resisting torque T_1 and T_2 at the other end of the shafts, then the shafts are said to be connected in parallel as shown

In such case, the angle of twist is same for both shafts



$$\theta_1 = \theta_2$$

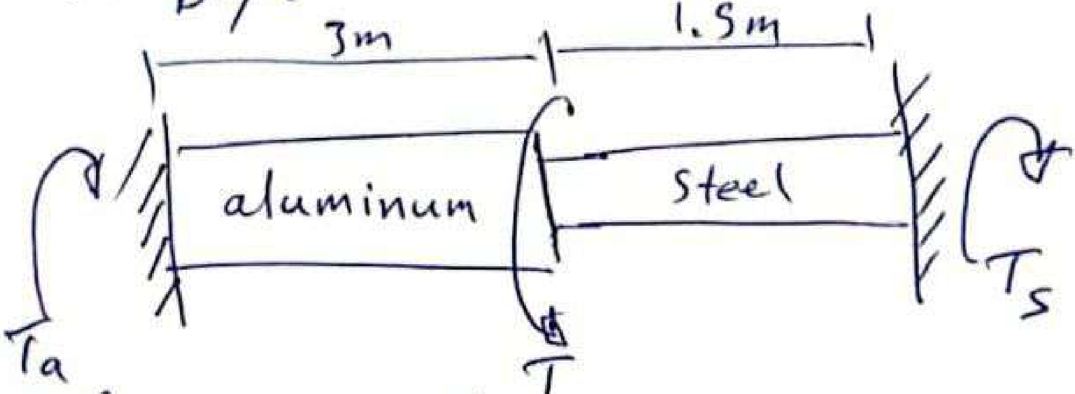
$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \Rightarrow \frac{T_1}{T_2} = \frac{G_1}{G_2} \times \frac{J_1}{J_2} \times \frac{L_2}{L_1}$$

if the shaft are made of the same material

$$G_1 = G_2$$

$$\Rightarrow \frac{T_1}{T_2} = \frac{J_1}{J_2} \times \frac{L_2}{L_1}$$

Ex- Two solid shafts of different materials are rigidly fastened together and attached to rigid supports as shown. The aluminum segment is 75 mm in diameter and $G_a = 28 \times 10^9$ N/m^2, the steel segment has a diameter of 50mm and $G_s = 83 \times 10^9$ N/m^2. The Torque, $T = 1000$ N.m is applied at the junction of two segments. Compute the shearing stress developed in the assembly.



The shaft in parallel

$$\theta_1 = \theta_2$$

$$\left(\frac{TL}{GJ}\right)_s = \left(\frac{TL}{GJ}\right)_a \Rightarrow \frac{T_s}{T_a} = \frac{G_s}{G_a} \times \frac{J_s}{J_a} \times \frac{L}{L}$$

$$\Rightarrow \frac{T_s}{T_a} = 1.17 \quad \Rightarrow T_s = 1.17\,T_a$$

for equilibrum $\Sigma M = 0 \quad T_s + T_a = T = 1000$

$\Rightarrow T_a + 1.17T_a = 1000 \Rightarrow 2.17T_a = 1000$

$\Rightarrow T_a = 461$ N.m and $T_s = 539$ N.m