Signal Processing

Fourier Transform

Fourier Transform

- "The Fourier transform is a mathematical operation with many applications in physics and engineering that expresses a mathematical function of time as a function of frequency, known as its frequency spectrum."

  - "For instance, the transform of a musical chord made up of pure notes (without overtones) expressed as amplitude as a function of time, is a mathematical representation of the amplitudes and phases of the individual notes that make it up."
  - "The function of time is often called the time domain representation, and the frequency spectrum the frequency domain representation."

Fourier Transform

Applications

- ❖ Differential equations
- ❖ Geology
- ❖ Image and signal processing
- ❖ Optics
- ❖ Quantum mechanics
- ❖ Spectroscopy

Fourier Transform

The ***Fourier transform*** **of a signal g(t) is defined by**

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

and g(t) is called the inverse Fourier transform of G(ω)

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega$$

The functions g(t) and G(ω) constitute a Fourier transform pair:

g(t) ⇔ G(ω)

G(ω) = F[g(t)] and g(t) = F^{-1}[G(ω)]

What is the difference between Fourier transform and Fourier series?

Fourier Transform

Fourier transform is different from the Fourier Series in that its frequency spectrum is continuous rather than discrete.

Fourier transform is obtained from Fourier series by letting $T \to \infty$ (for a nonperiodic signal).

The original time function can be uniquely recovered from its Fourier transform.

Fourier Transform and Fourier Series

- ***A periodic signal spectrum has finite amplitudes and exists at discrete set of frequencies.*** **Those amplitudes are also called the Fourier coefficients of the periodic signal**

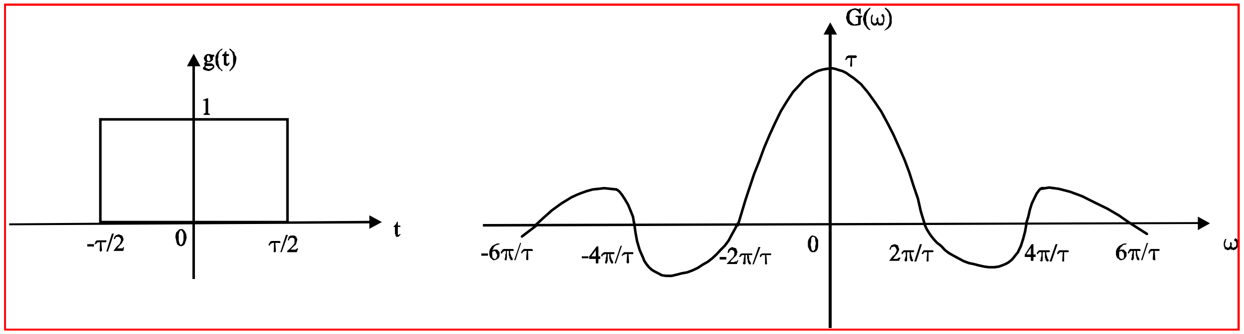
- ***A non-periodic signal has a continuous spectrum G(ω) and exist at all frequencies.***

Fourier transform of some useful functions

Rectangular function:

$$rect(\frac{t}{\tau}) = \begin{cases} 1 & -\frac{\tau}{2} < t < \frac{\tau}{2} \\ 0 & elsewhere \end{cases}$$

$$rect(\frac{t}{\tau}) \Leftrightarrow \tau Sa(\frac{\omega\tau}{2})$$

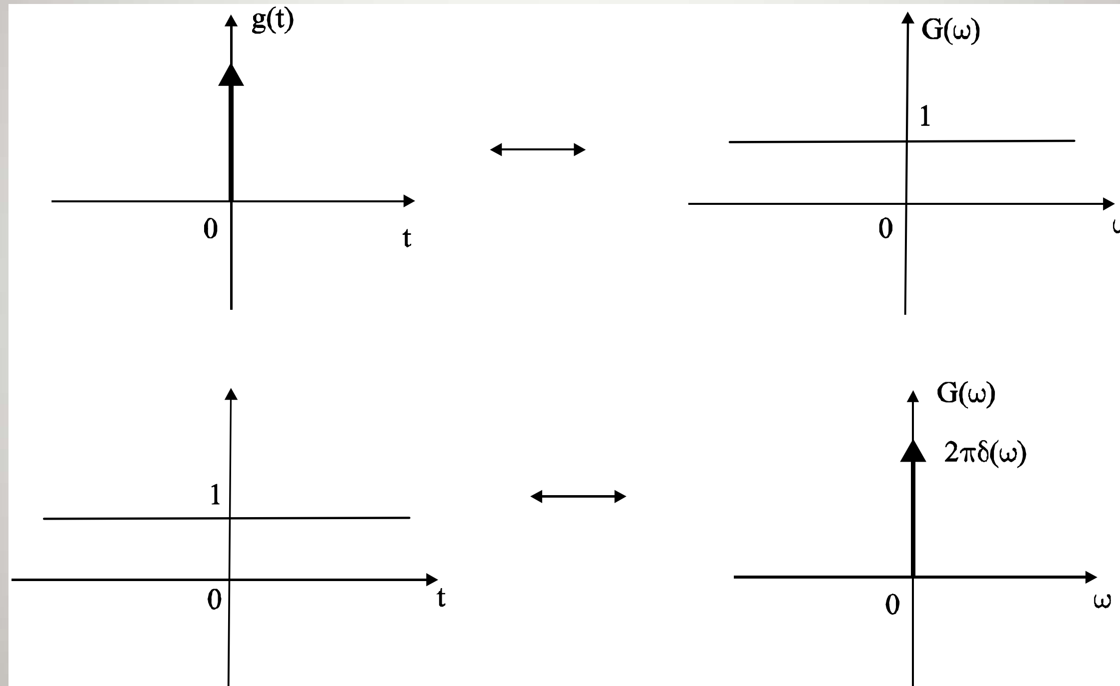


Proof

$$G(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = \frac{e^{-j\omega t}}{-j\omega} \Big|_{-\tau/2}^{\tau/2} = \frac{e^{-j\omega\tau/2} - e^{j\omega\tau/2}}{-j\omega} = \tau \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega\tau} = \frac{2}{\omega}\sin\frac{\omega\tau}{2} = \tau Sa(\frac{\omega\tau}{2})$$

Fourier transform of some useful functions

Unit impulse function: $\delta(t) \Leftrightarrow 1$ and $1 \Leftrightarrow 2\pi\delta(\omega)$

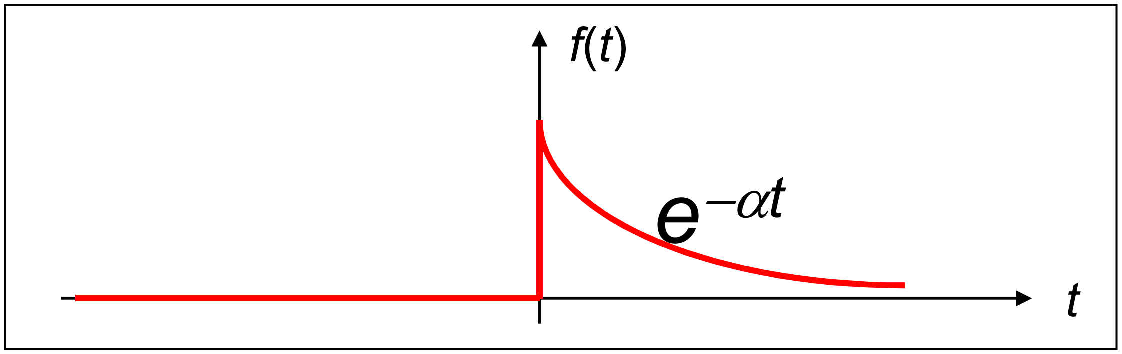


Proof

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{0} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

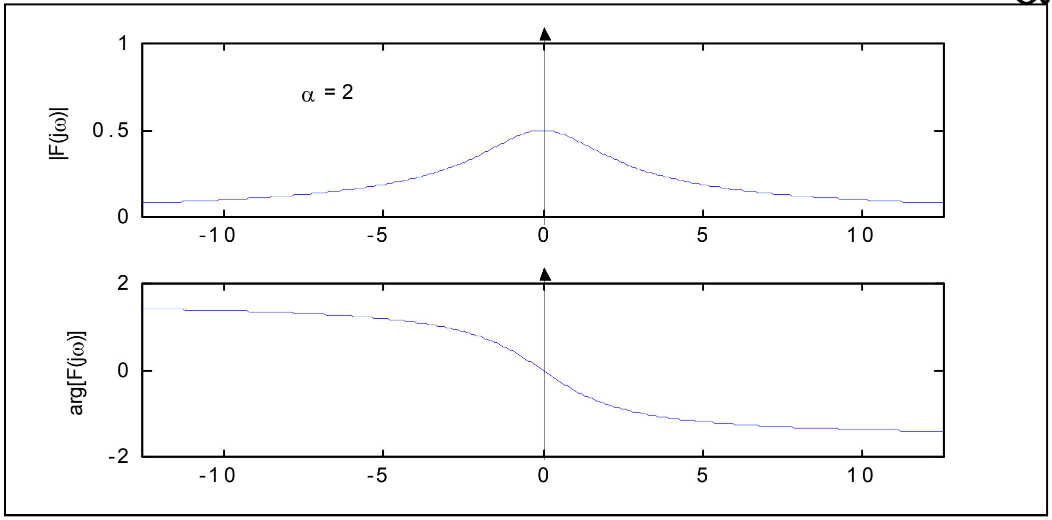
$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}$$

$$f(t) = e^{-\alpha} u(t)$$



$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad = \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(\alpha + j\omega)t} dt \quad = \frac{1}{\alpha + j\omega}$$



Example 1

$$e(t)=\begin{cases} e^{at}, & t<0 \\ e^{-at}, & t\geq 0 \end{cases}$$

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

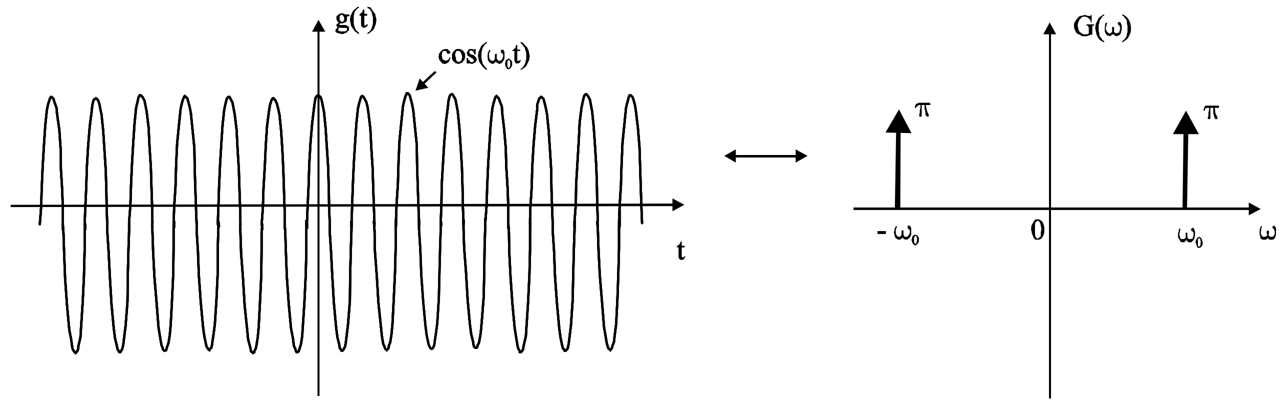
$$=\int_{-\infty}^{0} e^{at}e^{-j\omega t}dt+\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$=\frac{1}{(a-j\omega)}+\frac{1}{(a+j\omega)}=\frac{2a}{a^2+\omega^2}$$

Fourier transform of some useful functions

Sinusoidal function $\cos(\omega_0 t)$

$$\cos(\omega_0 t) \Leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$



Proof

$$F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

$$\therefore e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0), \qquad e^{-j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega+\omega_0)$$

$$\cos\omega_0 t \Leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

Properties of Fourier Transform

Linearity property

If $g_1(t) \Leftrightarrow G_1(\omega)$ **and** $g_2(t) \Leftrightarrow G_2(\omega)$

then $a_1 g_1(t) + a_2 g_2(t) \Leftrightarrow a_1 G_1(\omega) + a_2 G_2(\omega)$

where a_1 **and** a_2 **are constants**

This property is proved easily by linearity property of integrals used in defining Fourier transform

Properties of Fourier Transform

Symmetry property

If $g(t) \Leftrightarrow G(\omega)$, then $G(t) \Leftrightarrow 2\pi g(-\omega)$

Proof

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega$$

$$2\pi g(-t) = \int_{-\infty}^{\infty} G(\omega)e^{-j\omega t}d\omega$$

we can interchange the variable t and ω, i.e. let $t \to \omega$, $\omega \to t$, then

$$2\pi g(-\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t}dt$$

$$\therefore G(t) \Leftrightarrow 2\pi g(-\omega)$$

Properties of Fourier Transform

Time scaling property

$$g(at) \Leftrightarrow \frac{1}{|a|} G(\frac{\omega}{a})$$

Proof

$$F[g(at)] = \int_{-\infty}^{\infty} g(at) e^{-j\omega t} dt$$

let x = at, then dt = dx/a,

case 1: when a > 0,

$$F[g(at)] = \frac{1}{a} \int_{-\infty}^{\infty} g(x) e^{-j\omega x/a} dx = \frac{1}{a} G(\frac{\omega}{a})$$

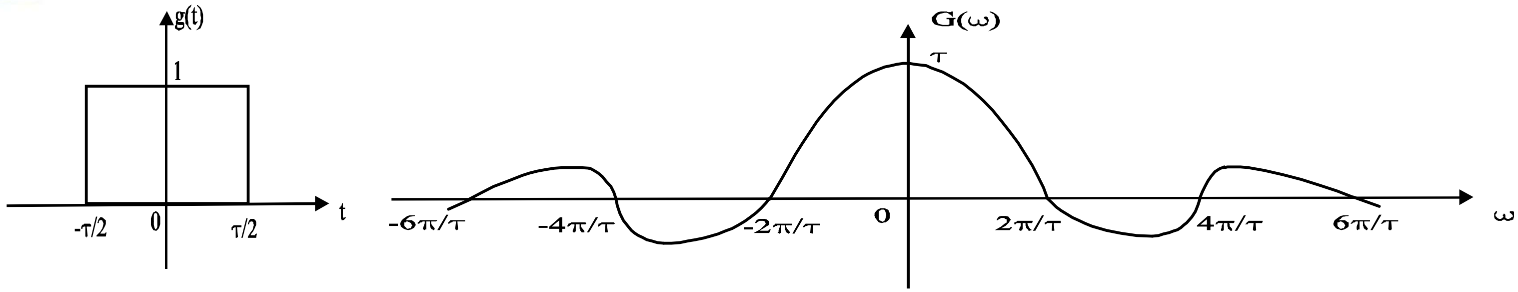
case 2: when a < 0, then t → ∞ leads to x → - ∞,

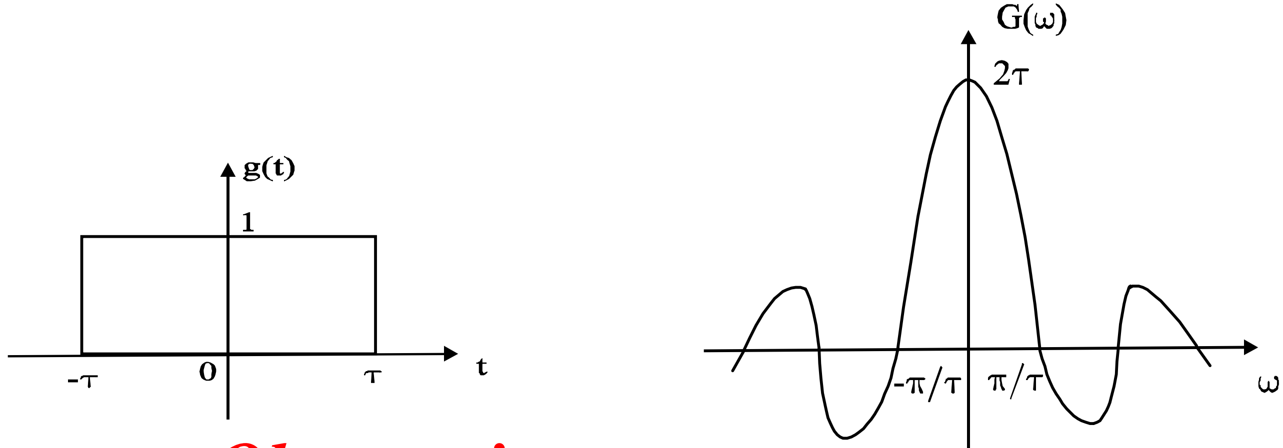
$$F[g(at)] = \frac{1}{a} \int_{\infty}^{-\infty} g(x) e^{-j\omega x/a} dx = -\frac{1}{a} \int_{-\infty}^{\infty} g(x) e^{-j\omega x/a} dx = -\frac{1}{a} G(\frac{\omega}{a})$$

Combined, the two cases are expressed as,

$$g(at) \Leftrightarrow \frac{1}{|a|} G(\frac{\omega}{a})$$

Properties of Fourier Transform





Important Observation:

Time domain ***compression*** *of a signal results in spectral* ***expansion***

Time domain ***expansion*** *of a signal results in spectral* ***compression***

Properties of Fourier Transform

Time shifting property

$$g(t-t_0) \Leftrightarrow G(\omega)e^{-j\omega t_0}$$

Proof

$$F[g(t-t_0)] = \int_{-\infty}^{\infty} g(t-t_0)e^{-j\omega t}dt$$

put t – t_0 = x, so that dt = dx, then

$$F[g(t-t_0)] = \int_{-\infty}^{\infty} g(x)e^{-j\omega(x+t_0)}dx = e^{-j\omega t_0}\int_{-\infty}^{\infty} g(x)e^{-j\omega x}dx = G(\omega)e^{-j\omega t_0}$$

Frequency shifting property

$$g(t)e^{j\omega_0 t} \Leftrightarrow G(\omega-\omega_0)$$

Proof

$$F[g(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}e^{j\omega_0 t}dt = \int_{-\infty}^{\infty} g(t)e^{-j(\omega-\omega_0)t}dt = G(\omega-\omega_0)$$

Properties of Fourier Transform

Significance

- ***Multiplication of a function g(t) by exp($j\omega_0 t$) is equivalent to shifting its Fourier transform in the positive direction by an amount ω_0.*** **--** ***Frequency translation theorem*.**

- ***Translation of a spectrum*** **helps in achieving *modulation*, which is performed by multiplying the known signal g(t) by a sinusoidal signal.**

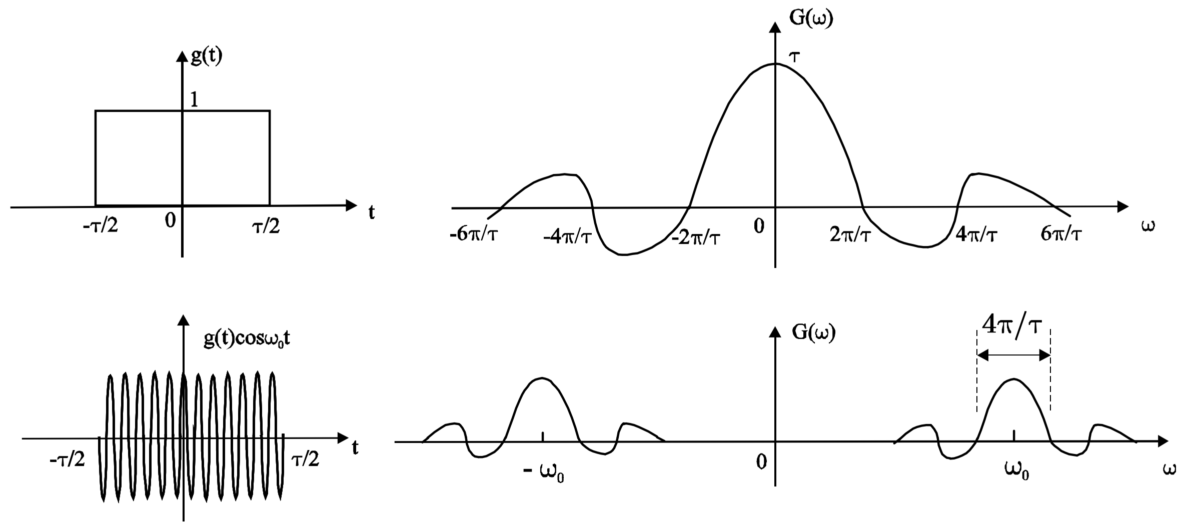
$$g(t)\cos\omega_0 t = \frac{1}{2}[g(t)e^{j\omega_0 t} + g(t)e^{-j\omega_0 t}]$$

Therefore,

$$g(t)\cos\omega_0 t \Leftrightarrow \frac{1}{2}[G(\omega-\omega_0) + G(\omega+\omega_0)]$$

Modulation Theorem

- The multiplication of a time function with a sinusoidal function translates the whole spectrum G(ω) to $\pm\omega_0$.
- $\exp(j\omega_0 t)$ can also provide frequency translation, but it is not a real signal. Hence, sinusoidal function is used in practical modulation system.



Properties of Fourier Transform

Convolution

Suppose that $g_1(t) \Leftrightarrow G_1(\omega)$ and $g_2(t) \Leftrightarrow G_2(\omega)$, then,

what is the waveform of g(t) whose Fourier transform is the product of $G_1(\omega)$ and $G_2(\omega)$?

This question arises frequently in spectral analysis, and is answered by the *convolution theorem*.

The convolution of two time function $g_1(t)$ and $g_2(t)$, is defined by the following integral

$$g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau$$

Convolution Theorem

Time convolution theorem

If $g_1(t) \Leftrightarrow G_1(\omega)$ and $g_2(t) \Leftrightarrow G_2(\omega)$

Then $g_1(t) * g_2(t) \Leftrightarrow G_1(\omega)G_2(\omega)$

$$F[g_1(t) * g_2(t)] = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} g_1(\tau)[\int_{-\infty}^{\infty} g_2(t-\tau)e^{-j\omega(t-\tau)}dt]e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} g_1(\tau)G_2(\omega)e^{-j\omega\tau}d\tau$$

$$= G_1(\omega)G_2(\omega)$$

Frequency convolution theorem

If $g_1(t) \Leftrightarrow G_1(\omega)$ and $g_2(t) \Leftrightarrow G_2(\omega)$

Then

$$g_1(t)g_2(t) \Leftrightarrow \frac{1}{2\pi}G_1(\omega) * G_2(\omega)$$

The proof is similar to time convolution theorem.

Convolution Theorem: Applications

$$g_1(t) * g_2(t) \Leftrightarrow G_1(\omega)G_2(\omega)$$

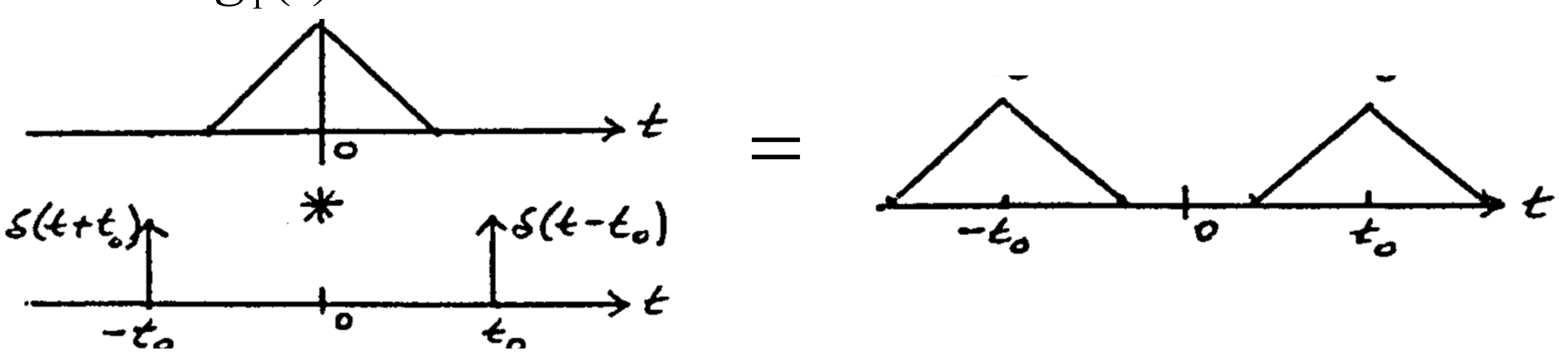
If we let $g_2(t) = \delta(t - t_0)$, then $g_1(t) * \delta(t - t_0) \Leftrightarrow G_1(\omega)e^{-j\omega t_0}$

But

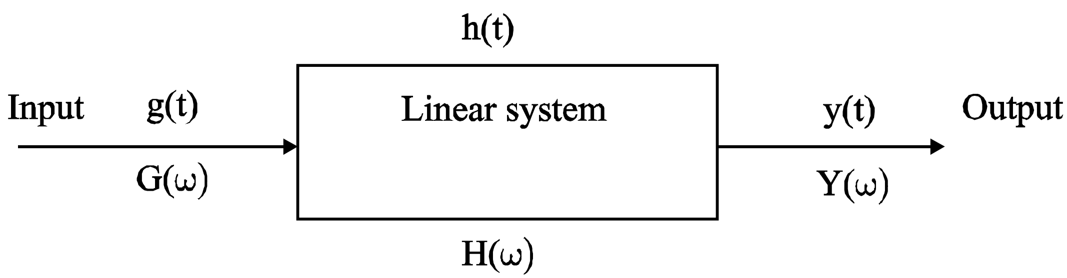
$$G_1(\omega)e^{-j\omega t_0} \Leftrightarrow g_1(t - t_0) \qquad \text{(time shifting property)}$$

Therefore, convolving with a delta function shifted in time by t_0 **corresponds to a shift of the original signal by** t_0

$g_1(t)$



Signal transmission through a linear system



Block diagram of a system

$$y(t) = g(t) * h(t)$$

when $g(t) \Leftrightarrow G(\omega)$, $h(t) \Leftrightarrow H(\omega)$, $y(t) \Leftrightarrow Y(\omega)$, $h(t)$ is the impulse response, i.e. if the input is $\delta(t)$, then $y(t) = h(t)$.

By convolution theorem

$$Y(\omega) = G(\omega)H(\omega)$$

where $H(\omega)$ is the system transfer function.

Signal Analysis

Signal power

- Signal-to-noise ratio (S/N) is an important parameter used to evaluate the system performance.
- Noise, being random in nature, cannot be expressed as a time function, like deterministic waveform. It is represented by power.

Hence, to evaluate the S/N, it is necessary to evolve a method for calculating the signal power.

For a general time domain signal g(t), its *average power* is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \, dt$$

Signal Analysis

For a periodic signal, each period contains a replica of the function, and the limiting operation can be omitted as long as T is taken as the period.

For a real signal

$$P = \overline{g^2(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)dt$$

Example

Find the power of a sinusoidal signal $\cos\omega_0 t$.

Solution

$$P = \overline{\cos^2(\omega_0 t)} = \frac{1}{T} \int_{-T/2}^{T/2} \frac{1+\cos 2\omega_0 t}{2} dt = \frac{1}{2T}(t + \frac{\sin 2\omega_0 t}{2\omega_0})\Big|_{-T/2}^{T/2} = \frac{1}{2}$$

Is it also possible to determine the signal power in frequency domain?

Signal Analysis

Frequency domain representation for signals of arbitrary waveshape

When dealing with *deterministic* signals, knowledge of the spectrum implies knowledge of the time domain signal.

For an arbitrary (*random*) signal, Fourier analysis cannot be used because g(t) is not known *analytically*.

For such an undeterministic signal (which include information signals and noise waveforms), the power spectrum $S_g(\omega)$ (or power spectral density) concept is used.

Signal Analysis

The power spectrum describes the distribution of power versus frequency.

The average signal power is then given by

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_g(\omega)d\omega = \frac{1}{\pi}\int_{0}^{\infty} S_g(\omega)d\omega$$

where $S_g(\omega) > 0$ for all ω.

Another way to evaluate the signal power!

Signal Analysis

Correlation

***Correlation* measure of similarity between one waveform, and time delayed version of the other waveform.**

The autocorrelation function is a special case of convolution, and it measures the similarity of a function with its delayed replica, and is given by

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) g^*(t+\tau) dt$$

Signal Analysis

Important properties of autocorrelation

(1) the *autocorrelation for $\tau = 0$* is *average power of the signal*

$$R(0) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} g(t)g^*(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2\,dt = P$$

The third way to evaluate signal power!

(2) power spectral density $S_g(\omega)$ and autocorrelation function of a power signal are Fourier transform pair

$$R(\tau) \Leftrightarrow S_g(\omega)$$

Exercise Problems (Signal Analysis)

1. Evaluate the integrals

$$\int_{-\infty}^{\infty} e^{\cos t}\delta(t-\pi)dt \qquad \int_{1}^{\infty} e^{-2x}\delta(x)dx \qquad \int_{-\infty}^{\infty} e^{-t}\delta(t+3)dt$$

$$\int_{-\infty}^{\infty} \delta(2t-4)(2t^2+t-8)dt \qquad \int_{-\infty}^{\infty} \cos(9t)\delta(t-2)dt$$

2. Simplify the following expressions:
(a) $[\sin t/(t+2)]\,\delta(t)$; (b) $[1/(j\omega+2)]\,\delta(\omega+3)$;
(c) $[\sin(k\omega)/\omega]\,\delta(\omega)$;

3. Calculate the (a) average value, (b) ac power, and (c) average power of the periodic waveform $v(t) = 1 + \cos\omega_0 t$.

Exercise Problems (Signal Analysis)

4. Prove that $$\delta(at) = \frac{1}{|a|}\delta(t)$$

5. If g(t) ⇔ G(ω), then show that g*(t) ⇔ G*(-ω).

6. Find the Fourier transform of the signal $f(t) = [A + f_m(t)]\cos\omega_c t$ if $f_m(t)$ has a spectrum $F_m(\omega)$.

7. If f(t) has a spectrum F(ω), find the Fourier transform of the following functions: (a) f(t/2 – 5); (b) f(3 – 3t); (c) f(2 + 5t);

8. Determine the average power of the following signals:
(a) $A\cos\omega_0 t + B\sin\omega_0 t$; (b) $(A + \sin\omega_0 t)\cos\omega_0 t$;

Math. Table

Properties of Fourier Transform

Linearity: $a_1 g_1(t) + a_2 g_2(t) \Leftrightarrow a_1 G_1(\omega) + a_2 G_2(\omega)$

Symmetry: If $g(t) \Leftrightarrow G(\omega)$, then $G(t) \Leftrightarrow 2\pi g(-\omega)$

Time scaling: $g(at) \Leftrightarrow \frac{1}{|a|} G(\frac{\omega}{a})$

Time shifting: $g(t - t_0) \Leftrightarrow G(\omega) e^{-j\omega t_0}$

Frequency shifting: $g(t) e^{j\omega_0 t} \Leftrightarrow G(\omega - \omega_0)$

Modulation theorem: $g(t)\cos\omega_0 t \Leftrightarrow \frac{1}{2}[G(\omega - \omega_0) + G(\omega + \omega_0)]$

Time convolution: $g_1(t) * g_2(t) \Leftrightarrow G_1(\omega) G_2(\omega)$

Frequency convolution: $g_1(t) g_2(t) \Leftrightarrow \frac{1}{2\pi} G_1(\omega) * G_2(\omega)$

Conjugate functions: $g^*(t) \Leftrightarrow G^*(-\omega)$

Time differentiation: $\frac{d}{dt} g(t) \Leftrightarrow j\omega G(\omega)$

Time integration: $\int_{-\infty}^{t} g(\tau) d\tau \Leftrightarrow \frac{1}{j\omega} G(\omega) + \pi G(0)\delta(\omega)$