Lecture Three
Methods of Analysis

3.1 Introduction

Having understood the fundamental laws of circuit theory (**Ohm's law** and **Kirchhoff's laws**), we are now prepared to apply these laws to develop two powerful techniques for circuit analysis: nodal analysis, which is based on a systematic application of Kirchhoff's current law (**KCL**), and mesh analysis, which is based on a systematic application of Kirchhoff's voltage law (**KVL**). The two techniques are so important that this chapter should be regarded as the most important in the lectures.

3.2 Nodal Analysis

Nodal analysis provides a general procedure for analyzing circuits using node voltages as the circuit variables. Choosing node voltages instead of element voltages as circuit variables is convenient and reduces the number of equations one must solve simultaneously. To simplify matters, we shall assume in this section that circuits do not contain voltage sources. Circuits that contain voltage sources will be analyzed in the next section.

Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign voltages v_1, v_2,. . v_{n-1} to the remaining **n-1** nodes. The voltages are referenced with respect to the reference node.
2. Apply **KCL** to each of the **n-1** nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

We shall now explain and apply these three steps.

The first step in nodal analysis is selecting a node as the reference or datum node. The reference node is commonly called the ground since it is assumed to have zero potential. A reference node is indicated by any of the three symbols in **Fig. 3.1**. We shall always use the symbol in **Fig. 3.1(b)**. Once we have selected a reference node, we assign voltage designations to nonreference nodes. Consider, for example, the circuit in **Fig. 3.2(a)**. Node 0 is the reference node (**v = 0**), while nodes 1 and 2 are assigned voltages v_1 and v_2, respectively. Keep in mind that the node voltages are defined with respect to the reference node. As illustrated in **Fig. 3.2(a)**, each node voltage is the voltage with respect to the reference node.

The number of nonreference nodes is equal to the number of independent equations that we will derive.

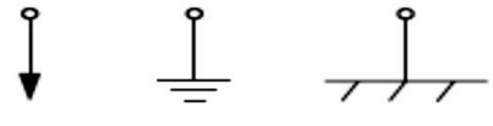


Figure 3.1 Common symbols for indicating a reference node.

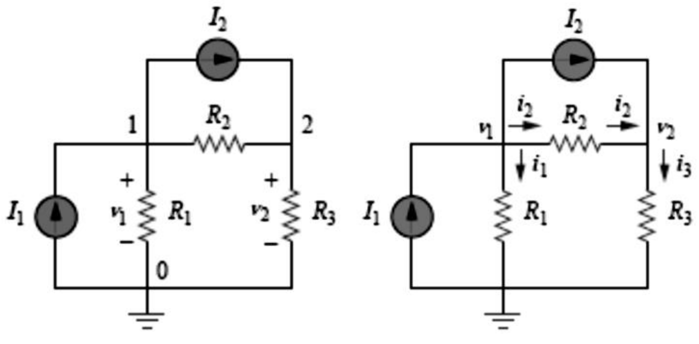


Figure 3.2 Typical circuits for nodal analysis.

As the second step, we apply **KCL** to each nonreference node in the circuit. To avoid putting too much information on the same circuit, the circuit in **Fig. 3.2(a)** is redrawn in **Fig. 3.2(b)**, where we now add $\mathbf{i_1}$, $\mathbf{i_2}$, and $\mathbf{i_3}$ as the currents through resistors $\mathbf{R_1}$, $\mathbf{R_2}$, and $\mathbf{R_3}$, respectively. At node 1, applying **KCL** gives

$$\mathbf{I_1 = I_2 + i_1 + i_2} \qquad (3.1)$$

At node 2,

$$\mathbf{I_2 + i_2 = i_3} \qquad (3.2)$$

We now apply Ohm's law to express the unknown currents $\mathbf{i_1}$, $\mathbf{i_2}$, and $\mathbf{i_3}$ in terms of node voltages.

Current flows from a higher potential to a lower potential in a resistor.

We can express this principle as

$$i = \frac{v_{higher} - v_{lower}}{R} \qquad (3.3)$$

Note that this principle is in agreement with the way we defined resistance in Chapter 2 (see **Fig. 2.3**). With this in mind, we obtain from **Fig. 3.2(b)**,

$$i_1 = \frac{v_1 - 0}{R_1}, \quad \text{or } i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2}, \quad \text{or } i_2 = G_2 (v_1 - v_2)$$

$$i_3 = \frac{v_2 - 0}{R_3}, \quad \text{or } i_3 = G_3 v_2 \qquad (3.4)$$

Substituting **Eq. (3.4) in Eqs. (3.1)** and **(3.2)** results, respectively, in

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} \qquad (3.5)$$

$$I_2 + v_1 - \frac{v_2}{R_2} = \frac{v_2}{R_3} \qquad (3.6)$$

In terms of the conductances, Eqs. (3.5) and (3.6) become

$$I_1 = I_2 + G_1 v_1 + G_2 (v_1 - v_2) \qquad (3.7)$$

$$I_2 + G_2 (v_1 - v_2) = G_3 v_2 \qquad (3.8)$$

The third step in nodal analysis is to solve for the node voltages. If we apply **KCL** to **n−1** nonreference nodes, we obtain **n−1** simultaneous equations such as **Eqs. (3.5)** and **(3.6)** or **(3.7)** and **(3.8)**. For the circuit of **Fig. 3.2**, we solve **Eqs. (3.5)** and **(3.6)** or **(3.7)** and **(3.8)** to obtain the node voltages v_1 and v_2 using any standard method, such as the substitution method, the elimination method, Cramer's rule, or matrix inversion. To use either of the last two methods, one must cast the simultaneous equations in matrix form. For example, **Eqs. (3.7)** and **(3.8)** can be cast in matrix form as

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix} \qquad (3.9)$$

which can be solved to get v_1 and v_2.

Example 3.1: Calculate the node voltages in the circuit shown in **Fig. 3.3(a)**.

Solution:

Consider **Fig. 3.3(b)**, where the circuit in **Fig. 3.3(a)** has been prepared for nodal analysis. Notice how the currents are selected for the application of **KCL**. Except for the branches with current sources, the labeling of the currents is arbitrary but consistent. (By consistent, we mean that if, for example, we assume that i2 enters the 4_resistor from the left-hand side, i2 must leave the resistor from the right-hand side.) The reference node is selected, and the node voltages v1 and v2 are now to be determined.

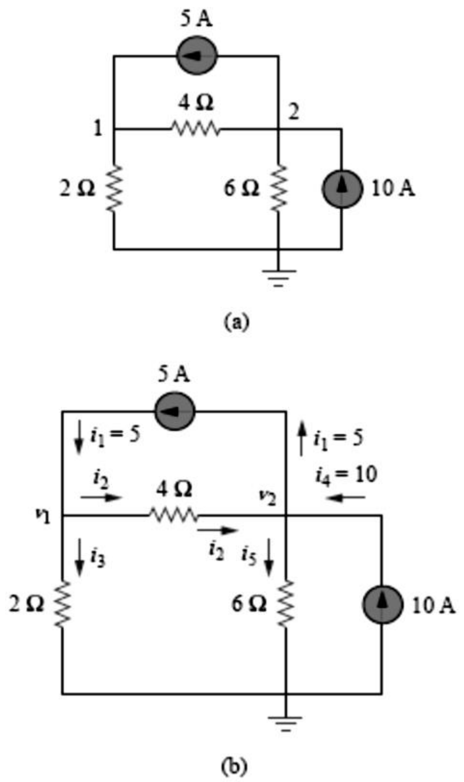


Figure 3.3 For Example 3.1: (a) original circuit, (b) circuit for analysis

At node 1, applying **KCL** and **Ohm's law** gives

$$i_1 = i_2 + i_3 \Rightarrow \mathbf{5} = \frac{v1 - v2}{4} + \frac{v1 - 0}{2}$$

Multiplying each term in the last equation by 4, we obtain

$$\mathbf{20 = v_1 - v_2 + 2v_1}$$

or

$$\mathbf{3v_1 - v_2 = 20} \qquad (3.1.1)$$

At node 2, we do the same thing and get

$$i_2 + i_4 = i_1 + i_5 \Rightarrow \frac{v1 - v2}{4} + \mathbf{10} = \mathbf{5} + \frac{v2 - 0}{6}$$

Multiplying each term by 12 results in

$$3v_1 - 3v= + 120 = 60 + 2v_2$$

or

$$-3v_1 + 5v_2 = 60 \qquad (3.1.2)$$

Now we have two simultaneous **Eqs. (3.1.1)** and **(3.1.2)**. We can solve the equations using any method and obtain the values of v1 and v2.

METHOD 1: Using the elimination technique, we add **Eqs. (3.1.1)** and **(3.1.2)**.

$$4v_2 = 80 \Rightarrow v_2 = 20 \text{ V}$$

Substituting $v_2 = 20$ in Eq. (3.1.1) gives

$$3v_1 - 20 = 20 \Rightarrow v_1 = 40/3 = 13.33 \text{ V}$$

METHOD 2: To use **Cramer's rule**, we need to put **Eqs. (3.1.1)** and **(3.1.2)** in matrix form as

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix} \qquad (3.1.3)$$

The determinant of the matrix is

$$\boldsymbol{\Delta = D = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 - 3 = 12}$$

We now obtain $\mathbf{v_1}$ and $\mathbf{v_2}$ as

$$\boldsymbol{v_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{D} = \frac{100+60}{12} = 13.33V}$$

$$\boldsymbol{v_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{D} = \frac{180+60}{12} = 20V}$$

If we need the currents, we can easily calculate them from the values of the nodal voltages. $\mathbf{i_1} = 5\ \mathbf{A}$, $i_2 = \frac{v_1 - v_2}{4} = -1.6667\ A$, $i_3 = \frac{v_1}{2} = 6.666A$, $\mathbf{i_4} = 10\ \mathbf{A}$, $i_5 = \frac{v_2}{6} = 3.333A$

The fact that $\mathbf{i_2}$ is negative shows that the current flows in the direction opposite to the one assumed.

3.2.1 Nodal Analysis with Voltage Sources

We now consider how voltage sources affect nodal analysis. We use the circuit in **Fig. 3.4** for illustration. Consider the following two possibilities.

CASE 1: If a voltage source is connected between the reference node and a nonreference node, we simply set the voltage at the nonreference node equal to the voltage of the voltage source. In **Fig. 3.4**, for example,

$$\mathbf{v_1 = 10\ V} \qquad (3.10)$$

Thus our analysis is somewhat simplified by this knowledge of the voltage at this node.

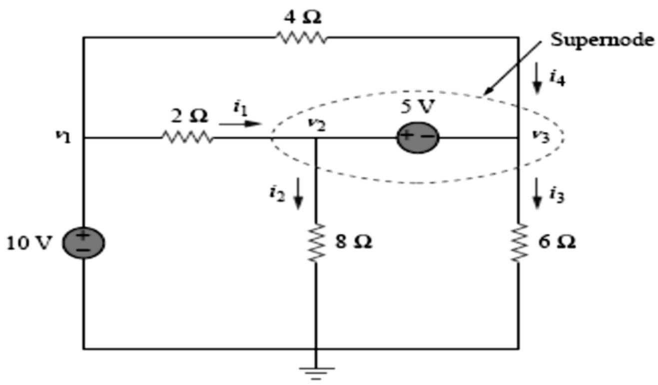


Figure 3.4 A circuit with a supernode.

CASE 2: If the voltage source (dependent or independent) is connected between two nonreference nodes, the two nonreference nodes form a generalized node or supernode; we apply both **KCL** and **KVL** to determine the node voltages.

A supernode is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

In **Fig. 3.4**, nodes 2 and 3 form a supernode. (We could have more than two nodes forming a single supernode. For example, see the circuit in the **Practice problem 3.4**). We analyze a circuit with supernodes using the same three steps mentioned in the previous section except that the supernodes are treated differently. Why? Because an essential component of nodal analysis is applying **KCL**, which requires knowing the current through each element. There is no way of knowing the current through a voltage source in advance. However, **KCL** must be satisfied at a supernode like any other node. Hence, at the supernode in **Fig. 3.5**,

$$\mathbf{i_1 + i_4 = i_2 + i_3} \quad (3.11a)$$

or

$$\frac{v1 - v2}{2} + \frac{v1 - v3}{4} = \frac{v2 - 0}{8} + \frac{v3 - 0}{6} \quad (3.11b)$$

To apply Kirchhoff's voltage law to the supernode in **Fig. 3.4**, we redraw the circuit as shown in **Fig. 3.5**. Going around the loop in the clockwise direction gives

$$\mathbf{-v_2 + 5 + v_3 = 0 \Rightarrow v_2 - v_3 = 5} \quad (3.12)$$

From **Eqs. (3.10), (3.11b),** and **(3.12)**, we obtain the node voltages.

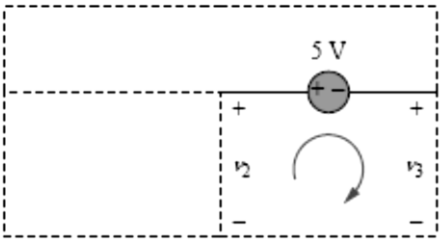


Figure 3.5 Applying KVL to a supernode.

Example 3.2: For the circuit shown in **Fig. 3.6**, find the node voltages.

Solution:

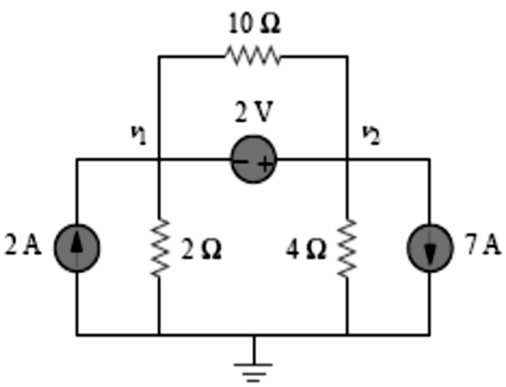


Figure 3.6 For Example 3.2.

The supernode contains the 2-V source, nodes 1 and 2, and the 10-Ω resistor. Applying **KCL** to the supernode as shown in **Fig. 3.7(a)** gives

$$\mathbf{2 = i_1 + i_2 + 7}$$

Expressing $\mathbf{i_1}$ and $\mathbf{i_2}$ in terms of the node voltages

$$\mathbf{2 = \frac{v1 - 0}{2} + \frac{v2 - 0}{4} + 7}$$

or

$$\mathbf{v_2 = -20 - 2v_1} \quad (3.2.1)$$

To get the relationship between $\mathbf{v_1}$ and $\mathbf{v_2}$, we apply **KVL** to the circuit in **Fig. 3.7(b)**. Going around the loop, we obtain

$$\mathbf{-v_1 - 2 + v_2 = 0 \;\Rightarrow v_2 = v_1 + 2} \quad (3.3.2)$$

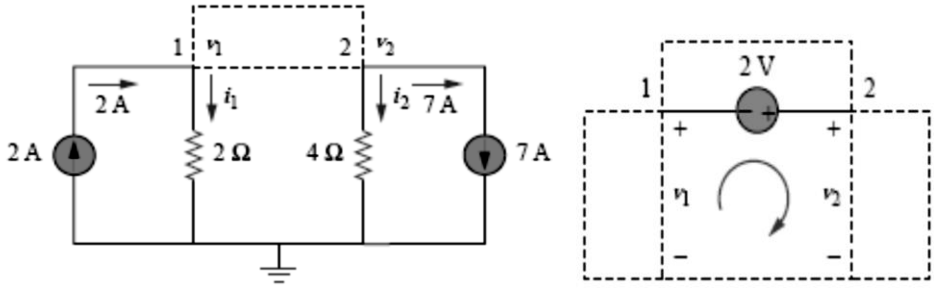
From **Eqs. (3.2.1)** and **(3.2.2)**, we write

$$\mathbf{v_2 = v_1 + 2 = -20 - 2v_1}$$

or

$$3v_1 = -22 \quad \Rightarrow v_1 = -7.333 \text{ V}$$

and $v_2 = v_1 + 2 = -5.333$ **V**. Note that the 10-Ω resistor does not make any difference because it is connected across the supernode.



(a) **(b)**

Figure 3.7 Applying: (a) KCL to the supernode, (b) KVL to the loop.

3.3 Mesh Analysis

Mesh analysis provides another general procedure for analyzing circuits, using mesh currents as the circuit variables. Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously. Recall that a loop is a closed path with no node passed more than once. A mesh is a loop that does not contain any other loop within it.

Nodal analysis applies **KCL** to find unknown voltages in a given circuit, while mesh analysis applies **KVL** to find unknown currents. Mesh analysis is not quite as general as nodal analysis because it is only applicable to a circuit that is planar. A **planar** circuit is one that can be drawn in a plane with no branches crossing one another; otherwise it is **nonplanar**. A circuit may have crossing branches and still be **planar** if it can be redrawn such that it has no crossing branches. For example, the circuit in **Fig. 3.8 (a)** has two crossing branches, but it can be redrawn as in **Fig. 3.8 (b)**. Hence, the circuit in **Fig. 3.8 (a)** is planar. However, the circuit in **Fig. 3.9** is **nonplanar**, because there is no way to redraw it and avoid the branches crossing. **Nonplanar** circuits can be handled using nodal analysis, but they will not be considered in this text.

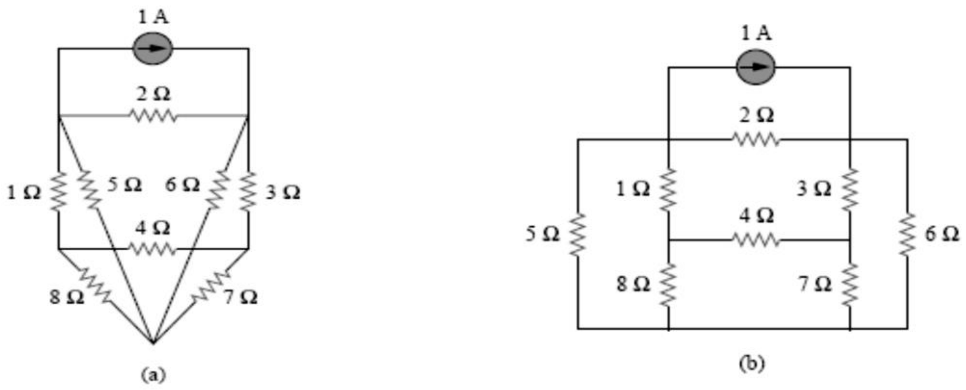


Figure 3.8 (a) A planar circuit with crossing branches, (b) the same circuit redrawn with no crossing branches.

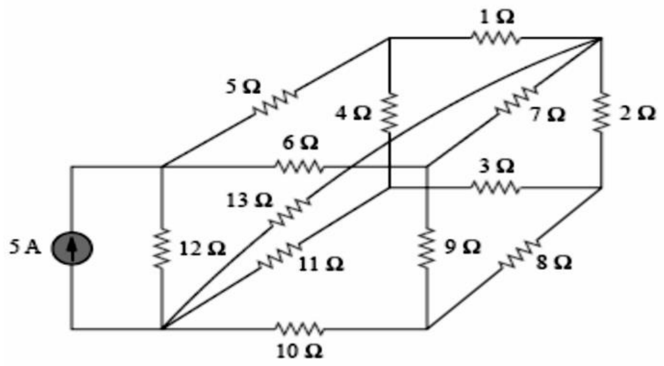


Figure 3.9 A nonplanar circuit.

To understand mesh analysis, we should first explain more about what we mean by a mesh. A mesh is a loop which does not contain any other loops within it.

In **Fig. 3.10**, for example, paths **abefa** and **bcdeb** are meshes, but path **abcdefa** is not a mesh. The current through a mesh is known as mesh current. In mesh analysis, we are interested in applying **KVL** to find the mesh currents in a given circuit.

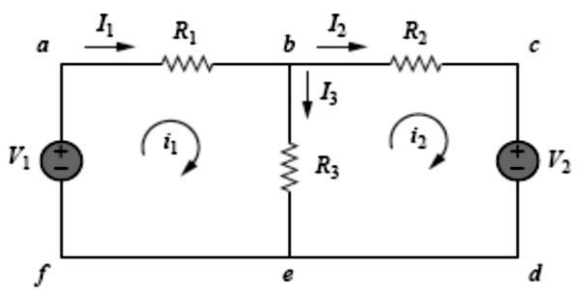


Figure 3.10 circuit with two meshes.

In this section, we will apply mesh analysis to planar circuits that do not contain current sources. In the next sections, we will consider circuits with current sources. In the mesh analysis of a circuit with **n** meshes, we take the following three steps.

Steps to Determine mesh currents:

1. Assign mesh currents $\mathbf{i_1}$, $\mathbf{i_2}$, . . . , $\mathbf{i_n}$ to the n meshes.
2. Apply **KVL** to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.
3. Solve the resulting n simultaneous equations to get the mesh currents.

To illustrate the steps, consider the circuit in **Fig. 3.10**. The first step requires that mesh currents $\mathbf{i_1}$ and $\mathbf{i_2}$ are assigned to meshes 1 and 2. Although a mesh current may be assigned to each mesh in an arbitrary direction, it is conventional to assume that each mesh current flows clockwise.

As the second step, we apply **KVL** to each mesh. Applying **KVL** to mesh 1, we obtain

$$-V_1 + R_1 i_1 + R_3 (i_1 - i_2) = 0$$

or

$$(R_1 + R_3)\, i1 - R_3 i_2 = V_1 \tag{3.13}$$

For mesh 2, applying KVL gives

$$R_2 i_2 + V_2 + R_3 (i_2 - i_1) = 0$$

or

$$-R_3 i_1 + (R_2 + R_3)\, i_2 = -V_2 \tag{3.14}$$

Note in **Eq. (3.13)** that the coefficient of $\mathbf{i_1}$ is the sum of the resistances in the first mesh, while the coefficient of $\mathbf{i_2}$ is the negative of the resistance common to meshes 1 and 2. Now observe that the same is true in **Eq. (3.14)**. This can serve as a shortcut way of writing the mesh equations.

The third step is to solve for the mesh currents. Putting **Eqs. (3.13)**. and **(3.14)** in matrix form yields

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ -V_2 \end{bmatrix} \tag{3.15}$$

which can be solved to obtain the mesh currents $\mathbf{i_1}$ and $\mathbf{i_2}$. We are at liberty to use any technique for solving the simultaneous equations. If a circuit has **n** nodes, **b** branches, and $\boldsymbol{l}$ independent loops or meshes, then $\boldsymbol{l = b - n + 1}$. Hence, $\boldsymbol{l}$ independent simultaneous equations are required to solve the circuit using mesh analysis.

Notice that the branch currents are different from the mesh currents unless the mesh is isolated. To distinguish between the two types of currents, we use **i** for a mesh current and **I** for a branch current. The current elements I_1, I_2, and I_3 are algebraic sums of the mesh currents. It is evident from **Fig. 3.13** that

$$I_1 = i_1, \quad I_2 = i_2, \quad I_3 = i_1 - i_2 \tag{3.16}$$

Example 3.3: For the circuit in **Fig. 3.11**, find the branch currents I_1, I_2, and I_3 using mesh analysis.

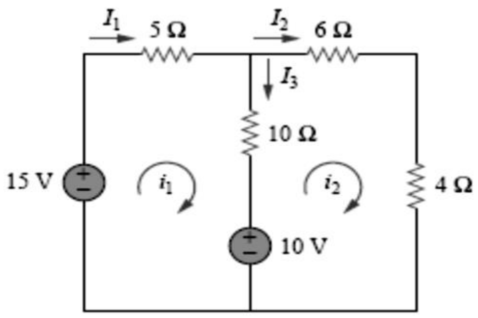


Figure 3.11 For Example 3.3.

Solution:

We first obtain the mesh currents using **KVL**. For mesh 1,

$$-15 + 5i_1 + 10(i_1 - i_2) + 10 = 0$$

or

$$3i_1 - 2i_2 = 1 \tag{3.5.1}$$

For mesh 2,

$$6i_2 + 4i_2 + 10(i_2 - i_1) - 10 = 0$$

or

$$i_1 = 2i_2 - 1 \tag{3.5.2}$$

Using the substitution method, we substitute **Eq. (3.3.2)** into **Eq. (3.3.1)**, and write

$$6i_2 - 3 - 2i_2 = 1 \Rightarrow \quad i_2 = 1\text{ A}$$

From **Eq. (3.5.2)**, $i_1 = 2i_2 - 1 = 2 - 1 = 1$ **A**. Thus,

$$I_1 = i_1 = 1\text{ A}, I_2 = i_2 = 1\text{ A}, I_3 = i_1 - i_2 = 0$$

3.3.1 Mesh Analysis with Current Sources

Applying mesh analysis to circuits containing current sources (dependent or independent) may appear complicated. But it is actually much easier than what we encountered in the previous section, because the presence of the current sources reduces the number of equations. Consider the following two possible cases.

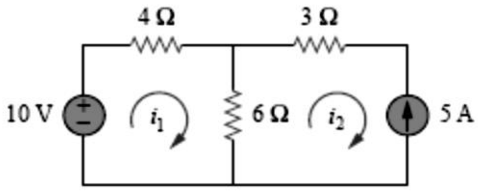


Figure 3.12 A circuit with a current source.

CASE 1: When a current source exists only in one mesh: Consider the circuit in **Fig. 3.12**, for example. We set $\mathbf{i_2} = -5$ A and write a mesh equation for the other mesh in the usual way, that is,

$$\mathbf{-10 + 4i_1 + 6(i_1 - i_2) = 0 \Rightarrow i_1 = -2\ A} \quad (3.17)$$

CASE 2: When a current source exists between two meshes: Consider the circuit in **Fig. 3.13(a)**, for example. We create a supermesh by excluding the current source and any elements connected in series with it, as shown in **Fig. 3.13(b)**. Thus,

A supermesh results when two meshes have a (dependent or independent) current source in common.

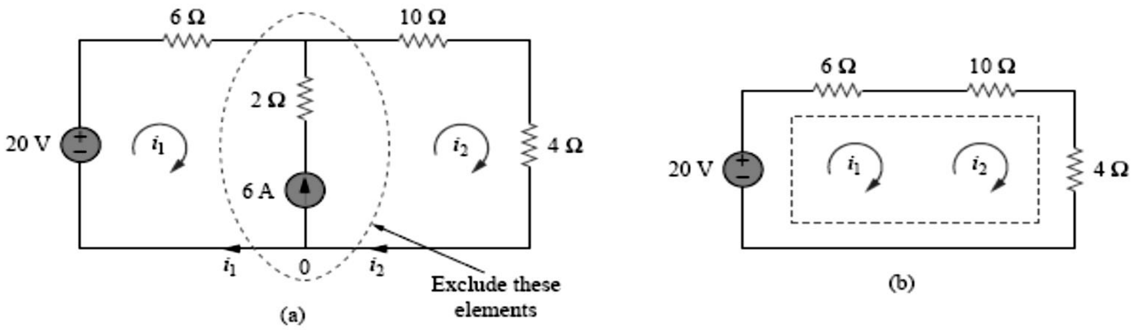


Figure 3.13 (a) Two meshes having a current source in common, (b) a supermesh, created by excluding the current source.

As shown in **Fig. 3.13(b)**, we create a supermesh as the periphery of the two meshes and treat it differently. (If a circuit has two or more supermeshes that intersect, they should be combined to form a larger supermesh.) Why treat the supermesh differently? Because mesh analysis applies **KVL**—which requires that we know the voltage across each branch—and we do not know the voltage across a current source in advance. However, a supermesh must satisfy **KVL** like any other mesh.

Therefore, applying **KVL** to the supermesh in **Fig. 3.13(b)** gives

$$\mathbf{-20 + 6i_1 + 10i_2 + 4i_2 = 0}$$

or

$$\mathbf{6i_1 + 14i_2 = 20} \quad (3.18)$$

We apply **KCL** to a node in the branch where the two meshes intersect.

Applying **KCL** to node 0 in **Fig. 3.13(a)** gives

$$\mathbf{i_2 = i_1 + 6} \tag{3.19}$$

Solving **Eqs. (3.18)** and **(3.19)**, we get

$$\mathbf{i_1 = -3.2\ A,\ i_2 = 2.8\ A} \tag{3.20}$$

Example 3.4: For the circuit in **Fig. 3.14**, find $\mathbf{i_1}$ to $\mathbf{i_4}$ using mesh analysis.

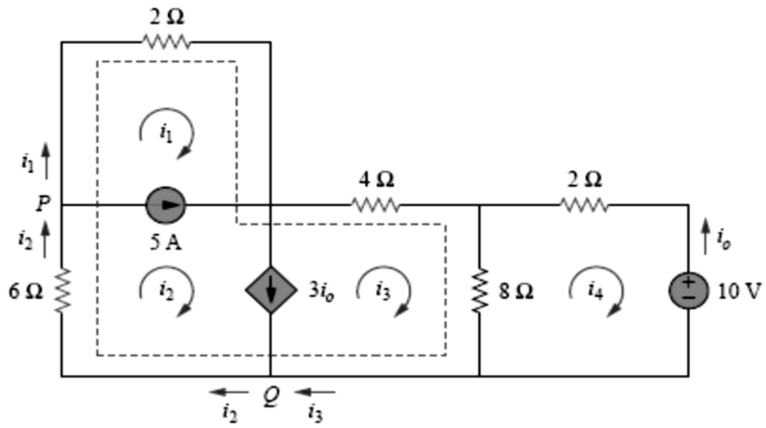


Figure 3.14 For Example 3.4.

Solution:

Note that meshes 1 and 2 form a supermesh since they have an independent current source in common. Also, meshes 2 and 3 form another supermesh because they have a dependent current source in common. The two supermeshes intersect and form a larger supermesh as shown. Applying **KVL** to the larger supermesh,

$$\mathbf{2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0}$$

or

$$\mathbf{i_1 + 3i_2 + 6i_3 - 4i_4 = 0} \tag{3.4.1}$$

For the independent current source, we apply **KCL** to node P:

$$\mathbf{i_2 = i_1 + 5} \tag{3.4.2}$$

For the dependent current source, we apply KCL to node Q:

$$\mathbf{i_2 = i_3 + 3i_o}$$

But $\mathbf{i_o = -i_4}$, hence,

$$\mathbf{i_2 = i_3 - 3i_4} \tag{3.4.3}$$

Applying **KVL** in mesh 4,

$$\mathbf{2i_4 + 8(i_4 - i_3) + 10 = 0}$$

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or

$$\mathbf{5i_4 - 4i_3 = -5} \qquad (3.4.4)$$

From **Eqs. (3.4.1)** to **(3.4.4)**,

$$\mathbf{i_1 = -7.5\ A,\ i_2 = -2.5\ A,\ i_3 = 3.93\ A,\ i_4 = 2.143\ A}$$