Poisson's Ratio: Biaxial and Triaxial Deformations

The ratio of strain in the lateral direction to the linear strain in the axial direction and it's denoted by the Greek letter υ (nu) and can be expressed as:

$$\upsilon = \frac{Lateral\ Strain}{Longitudinal\ Strain} = -\frac{\varepsilon y}{\varepsilon x}$$

υ : *Poison's Ratio*

$$\upsilon = -\frac{\varepsilon y}{\varepsilon x}$$

$$\varepsilon_y = -\upsilon \times \varepsilon_x = -\upsilon \frac{\sigma x}{E}$$

for biaxial stress state:

$\sigma_x \neq 0 \quad \sigma_y \neq 0 \ \sigma_z = 0$

In the x- direction resulting from
σ_x ,$\epsilon_x = \sigma_x / E$
In the y-direction resulting from
σ_y ,$\epsilon_y = \sigma_y / E$
In the x-direction resulting from
σ_y ,$\epsilon_x = -\upsilon(\sigma x E)$
In the y-direction resulting from the
σ_x , $\epsilon_y = -\upsilon(\sigma x E)$
The total strain in the x-direction will be:

$$\epsilon_x = \sigma_x / E - \upsilon\ \sigma_y / E$$

$$\epsilon_y = \sigma_y / E - \upsilon\ \sigma_x / E$$

$$\epsilon_z = -\upsilon\ \sigma_x / E - \upsilon\ \sigma_y / E$$

Triaxial tensile stresses:

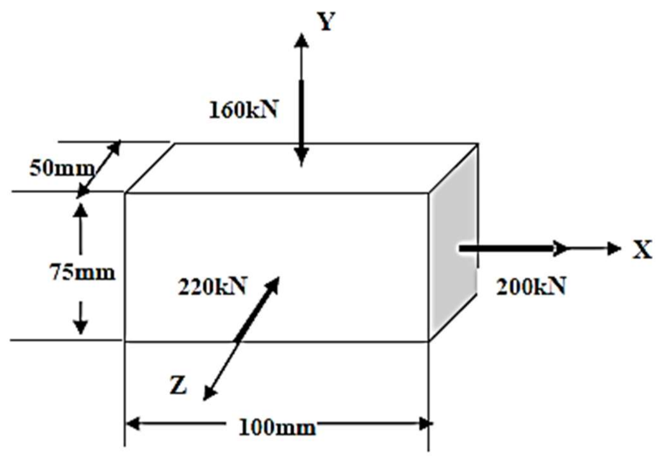
$\sigma_x \neq 0 \quad \sigma_y \neq 0 \; \sigma_z \neq 0$

$\epsilon_x = \sigma_x/E - \upsilon\, \sigma_y/E - \upsilon\, \sigma_z/E$

$\epsilon_y = \sigma_y/E - \upsilon\, \sigma_x/E - \upsilon\, \sigma_z/E$

$\epsilon_z = \sigma_z/E - \upsilon\, \sigma_x/E - \upsilon\, \sigma_y/E$

Ex: -9- A rectangular Aluminum block is (100mm) long in X-direction , (75mm) wide in Y-direction and (50mm) thick in Z-direction . It is subjected to try axial loading consisting of uniformly distributed tensile force of (200kN) in the X-direction and uniformly distributed compressive forces of 160kN in Y-direction and (220kN) in Z-direction. If the Poisson's ratio (υ =0.333) and (E=70GPa). Determine a single distributed load that must applied in Xdirection that would produce the same deformation in Z-direction as original loading.



$$\varepsilon_z = \frac{\sigma_z}{E} - \upsilon\frac{\sigma_y}{E} - \upsilon\frac{\sigma_x}{E}$$

$$\sigma_x = \frac{P_x}{A} = \frac{200*10^3}{0.05*0.075} = 53.3MPa\ (tension)$$

$$\sigma_y = \frac{160*10^3}{0.05*0.1} = 32MPa\ (compression)$$

$$\sigma_z = \frac{220*10^3}{0.075*0.1} = 24.34MPa\ (compression)$$

$$\therefore \varepsilon_z = \frac{1}{70*10^9}[-24.34 - 0.333(-32+53.3)]*10^6$$

$$\varepsilon_z = -0.52*10^{-3}$$

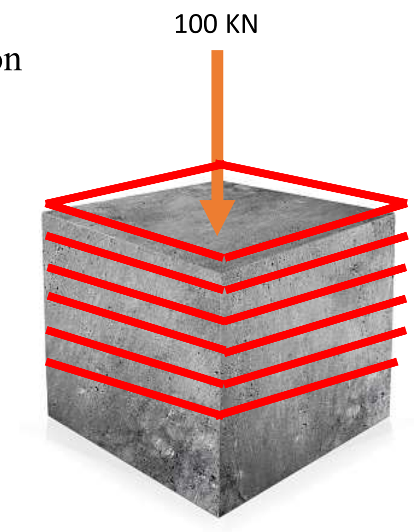
$$\varepsilon_z = -\upsilon\frac{\sigma_x}{E} = -\upsilon\frac{P_x}{A*E} \Rightarrow -0.52*10^{-3} = -0.333\frac{P_x}{0.075*0.05*70*10^9}$$

$$\therefore P_x = 410\ kN\ (Tension)$$

Example: a concrete cube of dimensions (150x150x150) mm is supported by rigid base and walls as shown in figure. Find the transverse stress and longitudinal deformation. use Ec=20 Gpa and υ =0.15.

Sol/

Since the cube is supported by walls in x and z direction



So

$\epsilon_x = \epsilon_z$

$\epsilon_x = \sigma_x/E - \upsilon\, \sigma_y/E - \upsilon\, \sigma_z/E$

$0 = \sigma_x/E - 0.15\, \sigma_y/E - 0.15\, \sigma_z/E$

$\sigma_{y= P/A} = \frac{-100\ X\ 10^3}{150\ X 15} = 0.677$

SO $\sigma_x = 0.15\, \sigma_Z - 0.677$ ……………1

$\epsilon_z = \sigma_z/E - \upsilon\, \sigma_x/E - \upsilon\, \sigma_y/E$

$0 = \sigma_z/E - 0.15\, \sigma_x/E - 0.15\, \sigma_y/E$

SO $\sigma_Z = 0.15\, \sigma_X - 0.677$ ……………2

By solving eq1 and 2 we get

$\sigma_x = -0.785$ MPA

$\sigma_z = -0.785$ MPA

SO:

$\epsilon_y = \sigma_y/E - \upsilon\, \sigma_x/E - \upsilon\, \sigma_z/E$

$\epsilon_y = -2.0144\ X\ 10^{-4}$

$\epsilon_y = \frac{\Delta L}{L} =$

$-2.0144\ X\ 10^{-4}$ X 150 = - 0.0316 MM (compression) OK

Deformation Of Axially Loaded Members

1- prismatic bodies

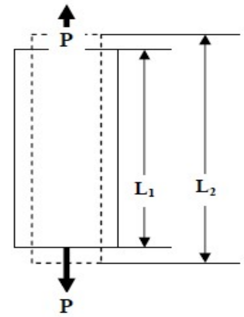
prismatic bar of homogenous materials loaded by constant force :

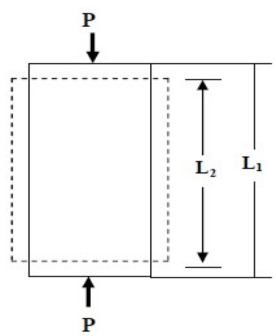
$\sigma = \frac{P}{A}$ ……………….1

$\epsilon = \frac{\Delta L}{L}$……………….2

$E = \frac{\sigma}{\varepsilon}$ ……….……3

$\Delta L = \frac{PL}{AE}$ …………… 4



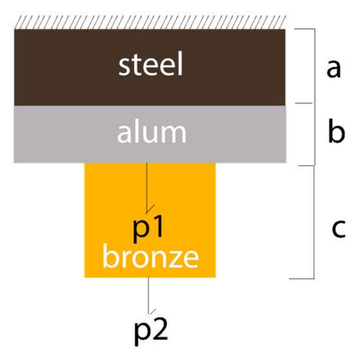


2- non prismatic bodies

non prismatic bar (different in dimensions) of non-homogenous materials loaded by multiple forces:

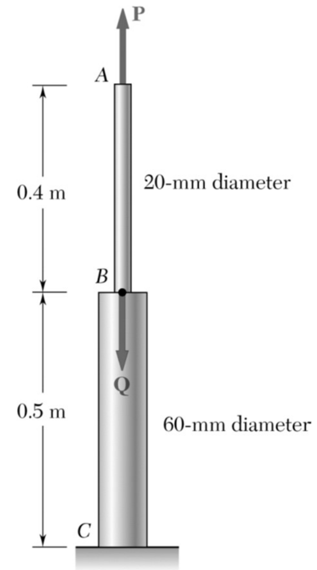
$\Delta L = \sum \frac{PL}{AE} = \Delta L1 + \Delta L2 + \Delta L3 + \Delta L4 + \Delta L5 + \ldots\ldots$

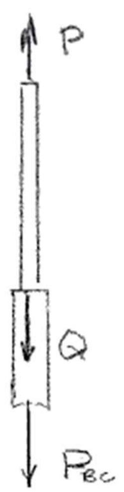
$= \frac{(P1+P2)a}{As\ Es} + \frac{(P1+p2)b}{Aa\ Ea} + \frac{P2\ c}{Ab\ Eb}$



Example:

The rod *ABC* is made of an aluminum for which $E = 70$ GPa. Knowing that P = 6 kN and Q = 42 kN, determine the deflection of (*a*) point *A*, (*b*) point *B*.





$$A_{AB} = \frac{\pi}{4} d_{AB}^2 = \frac{\pi}{4}(0.020)^2 = 314.16\times10^{-6}\,\text{m}^2$$

$$A_{BC} = \frac{\pi}{4} d_{BC}^2 = \frac{\pi}{4}(0.060)^2 = 2.8274\times10^{-3}\,\text{m}^2$$

$$P_{AB} = P = 6\times10^3\,\text{N}$$

$$P_{BC} = P - Q = 6\times10^3 - 42\times10^3 = -36\times10^3\,\text{N}$$

$$L_{AB} = 0.4\text{ m} \quad L_{BC} = 0.5\text{ m}$$

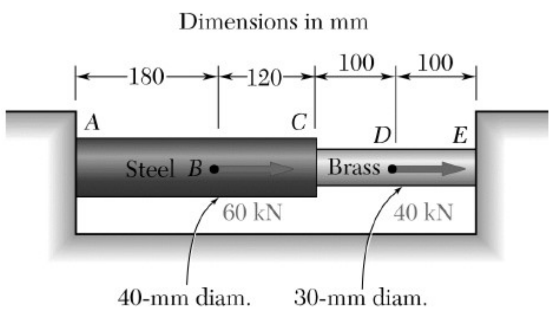
$$\delta_{AB} = \frac{P_{AB}L_{AB}}{A_{AB}E_A} = \frac{(6\times10^3)(0.4)}{(314.16\times10^{-6})(70\times10^9)} = 109.135\times10^{-6}\,\text{m}$$

$$\delta_{BC} = \frac{P_{BC}L_{BC}}{A_{BC}E} = \frac{(-36\times10^3)(0.5)}{(2.8274\times10^{-3})(70\times10^9)} = -90.947\times10^{-6}\,\text{m}$$

(*a*) $\delta_A = \delta_{AB} + \delta_{BC} = 109.135\times10^{-6} - 90.947\times10^{-6}\,\text{m} = 18.19\times10^{-6}\,\text{m}$ $\qquad \delta_A = 0.01819\text{ mm}\uparrow$ ◀

(*b*) $\delta_B = \delta_{BC} = -90.9\times10^{-6}\,\text{m} = -0.0909\text{ mm}$ $\qquad$ or $\qquad \delta_B = 0.0919\text{ mm}\downarrow$ ◀

Example:
Two cylindrical rods, one of steel and the other of brass, are joined at C and restrained by rigid supports at A and E. For the loading shown and knowing that Es = 200 GPa and Eb =105 GPa, determine (a) the reactions at A and E, (b) the deflection of point C.



SOLUTION

A to C: $E = 200 \times 10^9$ Pa

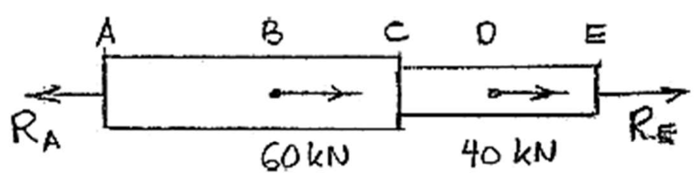
$$A = \frac{\pi}{4}(40)^2 = 1.25664 \times 10^3 \text{ mm}^2 = 1.25664 \times 10^{-3} \text{ m}^2$$

$$EA = 251.327 \times 10^6 \text{ N}$$

C to E: $E = 105 \times 10^9$ Pa

$$A = \frac{\pi}{4}(30)^2 = 706.86 \text{ mm}^2 = 706.86 \times 10^{-6} \text{m}^2$$

$$EA = 74.220 \times 10^6 \text{N}$$



A to B: $P = R_A$

$L = 180$ mm $= 0.180$ m

$$\delta_{AB} = \frac{PL}{EA} = \frac{R_A(0.180)}{251.327 \times 10^6}$$
$$= 716.20 \times 10^{-12} R_A$$

B to C: $P = R_A - 60 \times 10^3$

$L = 120$ mm $= 0.120$ m

$$\delta_{BC} = \frac{PL}{EA} = \frac{(R_A - 60 \times 10^3)(0.120)}{251.327 \times 10^6}$$
$$= 447.47 \times 10^{-12} R_A - 26.848 \times 10^{-6}$$

<u>*C* to *D*</u>: $P = R_A - 60\times10^3$

$L = 100 \text{ mm} = 0.100 \text{ m}$

$$\delta_{BC} = \frac{PL}{EA} = \frac{(R_A - 60\times10^3)(0.100)}{74.220\times10^6}$$

$$= 1.34735\times10^{-9} R_A - 80.841\times10^{-6}$$

<u>*D* to *E*</u>: $P = R_A - 100\times10^3$

$L = 100 \text{ mm} = 0.100 \text{ m}$

$$\delta_{DE} = \frac{PL}{EA} = \frac{(R_A - 100\times10^3)(0.100)}{74.220\times10^6}$$

$$= 1.34735\times10^{-9} R_A - 134.735\times10^{-6}$$

<u>*A* to *E*</u>: $\delta_{AE} = \delta_{AB} + \delta_{BC} + \delta_{CD} + \delta_{DE}$

$$= 3.85837\times10^{-9} R_A - 242.424\times10^{-6}$$

Since point *E* cannot move relative to *A*, $\delta_{AE} = 0$

(*a*) $3.85837\times10^{-9} R_A - 242.424\times10^{-6} = 0 \quad R_A = 62.831\times10^3 \text{N}$

$R_E = R_A - 100\times10^3 = 62.8\times10^3 - 100\times10^3 = -37.2\times10^3 \text{N}$

(*b*) $\delta_C = \delta_{AB} + \delta_{BC} = 1.16367\times10^{-9} R_A - 26.848\times10^{-6}$

$$= (1.16369\times10^{-9})(62.831\times10^3) - 26.848\times10^{-6}$$

$$= 46.3\times10^{-6} \text{m}$$