$$\bar{y}(s) = \frac{3s}{s^2+4} - \frac{10}{s^2+4} + \frac{4}{s^2}$$

$$y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{3s}{s^2+4} - \frac{10}{s^2+4} + \frac{4}{s^2}\right)$$

$$y = 3\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - 5\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) + 4\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$

$$y = 3\cos 2t - 5\sin 2t + 4t.$$

Example: Solve the equation

$$y' + 2y = e^t \quad , \quad y(0) = 1$$

$$\mathcal{L}(y') + \mathcal{L}(2y) = \mathcal{L}(e^t)$$

$$s\bar{y}(s) - y(0) + 2\bar{y}(s) = \frac{1}{s-1}$$

$$(s+2)\,\bar{y}(s) = \frac{1}{s-1} + 1$$

$$\bar{y}(s) = \frac{1}{(s-1)(s+2)} + \frac{1}{(s+2)}$$

$$\frac{1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} = \frac{A(s+2)+B(s-1)}{(s-1)(s+2)}$$

$$A(s+2) + B(s-1) = 1 \Rightarrow (A+B)s + 2A - B = 1$$

$$A+B=0 \quad \& \quad 2A - B = 1 \Rightarrow A = 1/3 \quad \& \quad B = -1/3$$

$$\bar{y}(s) = \frac{1/3}{s-1} - \frac{1/3}{s+2} + \frac{1}{s+2}$$

$$y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{1/3}{s-1} - \frac{1/3}{s+2} + \frac{1}{s+2}\right)$$

$$y = \frac{1}{3}e^t - \frac{1}{3}e^{-2t} + e^{-2t} \Rightarrow y = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}.$$

Example: Solve the equation

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^t.$$

For the boundary condition

$$\frac{d^2y}{dt^2}(0) = -2, \quad \frac{dy}{dt}(0) = 0, \quad y(0) = 1$$

$$\mathcal{L}\left(\frac{d^3y}{dt^3}\right) - \mathcal{L}\left(3\frac{d^2y}{dt^2}\right) + \mathcal{L}\left(3\frac{dy}{dt}\right) - \mathcal{L}(y) = \mathcal{L}(t^2e^t)$$

$$s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0) - 3(s^2\bar{y}(s) - sy(0) - y'(0)) + 3(s\bar{y}(s) - y(0))$$

$$- \bar{y}(s) = \frac{2!}{(s-1)^3}$$

$$s^3\bar{y}(s) - s^2 + 2 - 3(s^2\bar{y}(s) - s) + 3(s\bar{y}(s) - 1) - \bar{y}(s) = \frac{2!}{(s-1)^3}$$

$$(s^3 - 3s^2 + 3s - 1)\bar{y}(s) - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$

$$\bar{y}(s) = \frac{s^2 - 3s + 1}{s^3 - 3s^2 + 3s - 1} + \frac{2}{(s-1)^3(s^3 - 3s^2 + 3s - 1)}$$

$$\bar{y}(s) = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y}(s) = \frac{s^2 - 2s + 1 - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y}(s) = \frac{(s-1)^2 - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y}(s) = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y}(s) = \frac{(s-1)^2}{(s-1)^3} - \frac{s-1}{(s-1)^3} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}\right)$$

$$y = e^t - te^t - \frac{1}{2!}t^2e^t + \frac{2}{5!}t^5e^t$$

Example: Solve the equation

$$\frac{d^2y}{dt^2} + 9y = \cos 2t$$

for the condition

$$y\left(\frac{\pi}{2}\right) = -1 \quad \& \quad y(0) = 1$$

Let $y'(0) = c$

$$\mathcal{L}\left(\frac{d^2y}{dt^2}\right) + \mathcal{L}(9y) = \mathcal{L}(\cos 2t)$$

$$s^2\bar{y}(s) - s\,y(0) - y'(0) + 9\bar{y}(s) = \frac{s}{s^2+4}$$

$$s^2\bar{y}(s) - s - c + 9\bar{y}(s) = \frac{s}{s^2+4}$$

$$(s^2+9)\bar{y}(s) = s + c + \frac{s}{s^2+4}$$

$$\bar{y}(s) = \frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$(As+B)(s^2+9) + (Cs+D)(s^2+4) = s$$

$$As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs + Ds^2 + 4D = s$$

$$(A+C)s^3 + (B+D)s^2 + (9A+4C)s + (9B+4D) = s$$

$$A + C = 0 \Rightarrow A = -C$$

$$B + D = 0 \Rightarrow B = -D$$

$$9A + 4C = 1 \Rightarrow -9C + 4C = 1 \Rightarrow -5C = 1 \Rightarrow C = -1/5$$

$$9B + 4D = 0 \Rightarrow -9D + 4D = 0 \Rightarrow -5D = 0 \Rightarrow D = 0$$

$$A = 1/5 \;,\; B = 0 \;,\; C = -1/5 \;,\; D = 0$$

$$\bar{y}(s) = \frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}$$

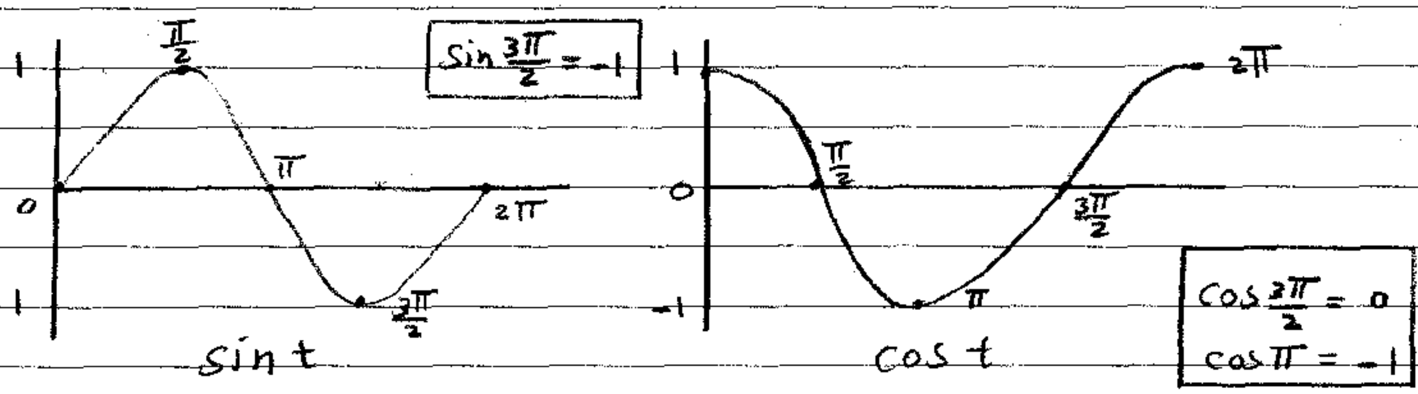
$$y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}\right)$$

$$y = \cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

To find $c \Rightarrow$ at $t = \frac{\pi}{2} \Rightarrow y = -1$

$$-1 = 0 - \frac{c}{3} - \frac{1}{5} \Rightarrow c = \frac{12}{5}$$

$$y = \cos 3t + \frac{12}{5}\sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t .$$



2. Ordinary Differential Equations with Variable Coefficients:

Example: Solve the equation

$$y'' - ty' + y = 1$$

For the Boundary Condition

$$y(0) = 1 \quad \& \quad y'(0) = 2$$

$$\mathcal{L}(y'') - \mathcal{L}(ty') + \mathcal{L}(y) = \mathcal{L}(1)$$

$$s^2\bar{y}(s) - sy(0) - y'(0) + \frac{d}{ds}\left(s\bar{y}(s) - y(0)\right) + \bar{y}(s) = \frac{1}{s}$$

$$* \quad \mathcal{L}\, x^n f(x) = (-1)^n \frac{d^n f(s)}{ds^n} \quad , \quad x = t \quad \& \quad n = 1$$

$$s^2\bar{y}(s) - s - 2 + s\bar{y}'(s) + \bar{y}(s) + \bar{y}(s) = \frac{1}{s}$$

$$s\bar{y}'(s) + (s^2 + 2)\bar{y}(s) = s + 2 + \frac{1}{s} \qquad \div s$$

$$\bar{y}'(s) + \left(s + \frac{2}{s}\right)\bar{y}(s) = 1 + \frac{2}{s} + \frac{1}{s^2}$$

$$\frac{d\bar{y}(s)}{ds} + \left(s + \frac{2}{s}\right)\bar{y}(s) = 1 + \frac{2}{s} + \frac{1}{s^2} \qquad \text{linear}$$

$$* \quad \frac{dy}{dx} + P(x)\,y = Q(x) \quad , \quad x = s$$

$$R = e^{\int\left(s + \frac{2}{s}\right)ds} = e^{\frac{1}{2}s^2 + 2\ln s} = s^2 e^{\frac{1}{2}s^2}$$

$$* \quad R.y = \int R.Q(x)\, dx + C$$

$$s^2 e^{\frac{1}{2}s^2}\,\bar{y}(s) = \int \left(1 + \frac{2}{s} + \frac{1}{s^2}\right) s^2 e^{\frac{1}{2}s^2}\, ds + C$$

$$\bar{y}(s) = \frac{1}{s^2} e^{-\frac{1}{2}s^2} \int \left(1 + \frac{2}{s} + \frac{1}{s^2}\right) s^2 e^{\frac{1}{2}s^2} \, ds + C$$

$$\bar{y}(s) = \frac{1}{s^2} e^{-\frac{1}{2}s^2} \int \left(s^2 e^{\frac{1}{2}s^2} + 2s\, e^{\frac{1}{2}s^2} + e^{\frac{1}{2}s^2}\right) ds + C$$

$$u = s \qquad du = ds$$

$$dv = s\, e^{\frac{1}{2}s^2} \, ds \qquad v = e^{\frac{1}{2}s^2}$$

$$\bar{y}(s) = \frac{1}{s^2} e^{-\frac{1}{2}s^2} \left(s\, e^{\frac{1}{2}s^2} \cancel{- \int e^{\frac{1}{2}s^2} ds} + 2 e^{\frac{1}{2}s^2} \cancel{+ \int e^{\frac{1}{2}s^2} ds}\right) + C$$

$$\bar{y}(s) = \frac{1}{s^2} e^{-\frac{1}{2}s^2} \left(s\, e^{\frac{1}{2}s^2} + 2 e^{\frac{1}{2}s^2} + C \right)$$

$$\bar{y}(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{C}{s^2} e^{-\frac{1}{2}s^2}$$

$$\bar{y}(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{C}{s^2}\left(1 - \frac{1}{2}s^2 + \frac{1}{8}s^4 - \cdots\right)$$

$$\bar{y}(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{C}{s^2} - \frac{C}{2} + \frac{C}{8}s^2 - \cdots$$

$$\bar{y}(s) = \frac{1}{s} + \frac{C+2}{s^2} - C\left(\frac{1}{2} - \frac{1}{8}s^2 + \cdots\right)$$

$* \quad \mathcal{L}^{-1}(s^x) = 0 \quad$ for $x = 0, 1, 2, 3, \ldots$

$$\mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{C+2}{s^2} - C\left(\frac{1}{2} - \frac{1}{8}s^2 + \cdots\right)\right)$$

$$y = 1 + (C+2)t \Rightarrow y' = 0 + C + 2$$

$$y'(0) = 2 \Rightarrow 2 = 0 + C + 2 \Rightarrow C = 0$$

$$\therefore \; y = 1 + 2t$$

Example: Solve the equation

$$ty'' - ty' + y = 0$$

For the boundary condition

$$y(0) = 0 \quad \& \quad y'(0) = 1$$

$$\mathcal{L}(ty'') - \mathcal{L}(ty') + \mathcal{L}(y) = 0$$

$$-\frac{d}{ds}(s^2\bar{y}(s) - sy(0) - y'(0)) + \frac{d}{ds}(s\bar{y}(s) - y(0)) + \bar{y}(s) = 0$$

$$-\frac{d}{ds}(s^2\bar{y}(s) - 1) + \frac{d}{ds}(s\bar{y}(s)) + \bar{y}(s) = 0$$

$$-s^2\bar{y}'(s) - 2s\bar{y}(s) + s\bar{y}'(s) + \bar{y}(s) + \bar{y}(s) = 0$$

$$(-s^2 + s)\bar{y}'(s) + (-2s + 1 + 1)\bar{y}(s) = 0$$

$$-s(s-1)\bar{y}'(s) - 2(s-1)\bar{y}(s) = 0$$

$$s\bar{y}'(s) + 2\bar{y}(s) = 0 \Rightarrow s\frac{d\bar{y}(s)}{ds} + 2\bar{y}(s) = 0$$

$$\frac{d\bar{y}(s)}{2\bar{y}(s)} = -\frac{ds}{s} \Rightarrow \frac{1}{2}\ln\bar{y}(s) = -\ln s + \ln c$$

$$\bar{y}(s) = \frac{k}{s^2} \Rightarrow y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{k}{s^2}\right) = kt$$

$$y = kt \Rightarrow y' = k \quad \& \quad y'(0) = 1 \Rightarrow k = 1$$

$$\therefore \; y = t.$$

Example: Solve the equation

$$ty'' - ty' + y = 0$$

For the boundary condition

$$y(0) = 0 \quad \& \quad y'(0) = 1$$

$$\mathcal{L}(ty'') - \mathcal{L}(ty') + \mathcal{L}(y) = 0$$

$$-\frac{d}{ds}\left(s^2\bar{y}(s) - sy(0) - y'(0)\right) + \frac{d}{ds}\left(s\bar{y}(s) - y(0)\right) + \bar{y}(s) = 0$$

$$-\frac{d}{ds}\left(s^2\bar{y}(s) - 1\right) + \frac{d}{ds}\left(s\bar{y}(s)\right) + \bar{y}(s) = 0$$

$$-s^2\bar{y}'(s) - 2s\bar{y}(s) + s\bar{y}'(s) + \bar{y}(s) + \bar{y}(s) = 0$$

$$(-s^2 + s)\,\bar{y}'(s) + (-2s + 1 + 1)\,\bar{y}(s) = 0$$

$$-s(s-1)\,\bar{y}'(s) - 2(s-1)\,\bar{y}(s) = 0$$

$$s\bar{y}'(s) + 2\bar{y}(s) = 0 \Rightarrow s\,\frac{d\bar{y}(s)}{ds} + 2\bar{y}(s) = 0$$

$$\frac{d\bar{y}(s)}{2\bar{y}(s)} = -\frac{ds}{s} \Rightarrow \frac{1}{2}\ln\bar{y}(s) = -\ln s + \ln c$$

$$\bar{y}(s) = \frac{k}{s^2} \Rightarrow y = \mathcal{L}^{-1}(\bar{y}(s)) = \mathcal{L}^{-1}\left(\frac{k}{s^2}\right) = kt$$

$$y = kt \Rightarrow y' = k \quad \& \quad y'(0) = 1 \Rightarrow k = 1$$

$$\therefore\ y = t.$$

Partial Differential Equations by Laplace Transformation:

Example: Solve the PDE by using:

1. Separation of variable method.
2. Laplace transform method.

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$$

For the boundary condition

i) $\theta(0,t) = 0$

ii) $\theta(1,t) = 0$

iii) $\theta(x,0) = 3\sin 2\pi x$

1. Separation of variable:

$$\theta(x,t) = e^{-\beta^2 t}(A\cos\beta x + B\sin\beta x)$$

B.C. 1 $\quad x = 0 \ , \ \theta = 0$

$$0 = e^{-\beta^2 t}(A(1) + B(0)) \Rightarrow A = 0$$

$$\theta(x,t) = B\, e^{-\beta^2 t} \sin\beta x$$

B.C. 2 $\quad x = 1 \ , \ \theta = 0$

$$0 = B\, e^{-\beta^2 t} \sin\beta(1) \Rightarrow \beta = n\pi$$

$$\theta(x,t) = B\, e^{-(n\pi)^2 t} \sin n\pi x$$

I.C. $\quad t = 0 \ , \ \theta = 3\sin 2\pi x$

$$3\sin 2\pi x = B \sin n\pi x$$

$$\therefore \; B = 3 \quad \& \quad n = 2$$

$$\theta(x,t) = 3\, e^{-4\pi^2 t} \sin 2\pi x$$

2. Laplace transform:

$$\mathcal{L}\, \frac{\partial \theta}{\partial t} = \mathcal{L}\, \frac{\partial^2 \theta}{\partial x^2}$$

$$s\bar{\theta}(s) - \theta(0) = \frac{d^2 \bar{\theta}(s)}{dx^2}$$

$$s\,\bar{\theta}(s) - 3\sin 2\pi x = \frac{d^2 \bar{\theta}(s)}{dx^2}$$

$$\frac{d^2 \bar{\theta}(s)}{dx^2} - s\,\bar{\theta}(s) = -3\sin 2\pi x$$

$$(D^2 - s)\,\bar{\theta}(s) = -3\sin 2\pi x$$

$$m^2 - s = 0 \Rightarrow m = \mp \sqrt{s}$$

$$\bar{\theta}(s)_c = C_1 e^{\sqrt{s}\,x} + C_2 e^{-\sqrt{s}\,x}$$

$$\bar{\theta}(s)_p = \frac{-3\sin 2\pi x}{D^2 - s}$$

$$y_p = \frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \sin(ax+b)$$

$$\bar{\theta}(s)_p = \frac{-3\sin 2\pi x}{-(2\pi)^2 - s}$$

$$\bar{\theta}(s)_p = \frac{3\sin 2\pi x}{4\pi^2 + s}$$

$$\bar{\theta}(s) = C_1 e^{\sqrt{s}\,x} + C_2 e^{-\sqrt{s}\,x} + \frac{3\sin 2\pi x}{s + 4\pi^2}$$

B.C.1 $\quad x = 0 \; , \; \theta = 0 \Rightarrow \bar{\theta}(s) = 0$

$$0 = C_1(1) + C_2(1) + 0 \quad , \quad \sin(0) = 0$$

$$\therefore \; C_1 = -C_2$$

B.C.2 $\quad x = 1 \; , \; \theta = 0 \Rightarrow \bar{\theta}(s) = 0$

$$0 = C_1 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}} + 0 \quad , \quad \sin(2\pi) = 0$$

$$0 = -C_2 e^{\sqrt{s}} + C_2 e^{-\sqrt{s}}$$

$$0 = C_2\left(-e^{\sqrt{s}} + e^{-\sqrt{s}}\right) \Rightarrow C_2 = 0$$

$$C_1 = -C_2 \Rightarrow C_1 = 0$$

$$\therefore \; \bar{\theta}(s) = \frac{3\sin 2\pi x}{s + 4\pi^2}$$

$$\mathcal{L}^{-1}\bar{\theta}(s) = 3\sin 2\pi x \, \mathcal{L}^{-1}\frac{1}{s + 4\pi^2}$$

$$\theta(x,t) = 3\sin 2\pi x \; e^{-4\pi^2 t}$$