LECTURE (1)

DIFFERENTIAL EQUATIONS
(PART ONE)

A differential equation is an equation that involves one or more derivatives. They are classified by:

1- **Type** (ordinary, partial).

2- **Order** (the highest order derivative that occurs in the equation).
3- **Degree** (the highest power of the highest order derivative).

If y is a function of x, where y is called the dependent variable and x is called the independent variable, thus, differential equation is a relation between x and y which includes at least one derivative of y with respect to x.

If the differential equation involves only a single independent variable , this derivative is called ORDINARY DERIVATIVE & the equation is called ORDINARY DIFFERENTIAL EQUATION (ODE).

If the differential equation involves two or more independent variables, this derivative is called PARTIAL DERIVATIVE & the equation is called PARTIAL DIFFERENTIAL EQUATION (PDE).

- **y= f(x,t)**

$$\frac{\partial 2y}{\partial t^2} = c^2 \left(\frac{\partial 2y}{\partial x^2}\right) \qquad (2^{nd} \text{ order ; } 1^{st} \text{ degree})$$

- **y= f(x)**

$$\frac{dy}{dx} = 3x + 5 \qquad (1^{st} \text{ order ; } 1^{st} \text{ degree})$$

$$\left(\frac{d3y}{dx^3}\right)^2 + \left(\frac{d2y}{dx^2}\right)^4 = 0 \quad (3^{rd} \text{ order ; } 2^{nd} \text{ degree})$$

$$5\,\frac{d3y}{dx^3} + \cos\frac{d2y}{dx^2} + 2xy = 0 \quad (3^{rd} \text{ order ; } 1^{st} \text{ degree})$$

SOLUTION OF DIFFERENTIAL EQUATIONS:

1- **GENERAL SOLUTION.**
2- **PARTICULAR SOLUTION.**

y = x + c   (**general solution**)
If y = 2 & x = 1, then
2 = 1+c ; c = 1
y= x+1   (**particular solution**)

The differential equation may be linear or non-linear depending on the presence of the dependent variable y and its derivatives in one term of the equation.

$\frac{d2y}{dx^2}$ + 4**x** $\frac{dy}{dx}$ + 2**y** = 0   (**linear equation**)

$\frac{d2y}{dx^2}$ + 4**y** $\frac{dy}{dx}$ + 2**y** = 0   (**non-linear equation**)

$\frac{d2y}{dx^2} + \sin y = 0$ (**non- linear equation** since it contains $\boldsymbol{\sin y}$ which is non-linear

The complexity of solving differential equations increases with the order.

1) SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS:

1. **Variable Separable Equation.**
2. **Homogenous Equation.**
3. **Exact Equation.**
4. **Linear Equation.**
5. **Bernoulli's Equation.**

1. Variable Separable Equation.

A first order Ordinary Differential Equation has the form:

F(x,y,,y`) = 0

In theory, at least, the method of algebra can be used to write it in the form:

y` = G(x,y).

If G(x,y) can be factored to give:

G(x,y)= Mx.Ny,

then the equation is called separable.
To solve the separable equation y` = Mx.Ny, we rewrite it in the form

f(y)y` = g(x).

Integrating both sides gives:

$$\int f(y)\ \ y`\,dx = \int g(x)dx$$

$$\int f(y)\ dy = \frac{dy}{dx}\,dx$$

Ex (1) Solve the equation $\mathbf{y\,\frac{dy}{dx} + x\,y^2 = x\ .}$

Solution:

$$y\,\frac{dy}{dx} + x\,y^2 - x = 0$$

$$y\,\frac{dy}{dx} + (y^2 - 1)\,x = 0$$

$$\left(\frac{y}{y^2-1}\right)\frac{dy}{dx} + \left(\frac{y^2-1}{y^2-1}\right)x = 0$$

$$\left(\frac{y}{y^2-1}\right)dy + x.dx = 0$$

$$\int \frac{y}{y^2-1})\,dy + \int x.\,dx = 0$$

$$\frac{1}{2}\ln(y^2-1) + \frac{x^2}{2} + c = 0$$

Ex (2) Solve the equation $\mathbf{\frac{dy}{dx} = (1+ y^2)\ e^x\ .}$

Solution:

$$\frac{dy}{1+y^2} = e^x\,dx$$

$$e^x\,dx - \frac{dy}{1+y^2} = 0$$

$$\int e^x\,dx - \int \frac{dy}{1+\ y^2} = 0$$

$$e^x - tan^{-1}\,y = c$$

$$y = tan(e^x - c)$$

Ex (3) Solve the equation $\frac{dy}{dx} = cos(x + y)$.

Solution:

Let u = x+y

$$\frac{du}{dx} = 1 + \frac{dy}{dx} \quad \rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} - 1$$

Sub. for $\frac{dy}{dx}$ in the main equation

$$\frac{du}{dx} - 1 = cos(u)$$

$$\frac{du}{dx} = cos(u) + 1 \quad \rightarrow \quad dx = \frac{du}{1+cos(u)}$$

$$\int \frac{(1-cos\, u)}{(1-cos\, u)(1+cos\, u)}\, du = \int dx$$

$$\int \frac{(1-cos\, u)}{(1-cos^2 u)}\, du = \int dx$$

$$\int \frac{1}{(sin^2 u)}\, du - \int \frac{cos\, u}{sin^2 u}\, du = \int dx$$

$$\int csc^2 u.\, du - \int cot\, u.\, csc\, u\, du = \int dx$$

$$- cot\, u + csc\, u = x + c \quad \rightarrow \quad - cot(x + y) + csc(x + y) = x + c$$

2- Homogenous Equation:

A(x,y) dx + B (x,y) dy = 0

where the functions A(x,y) & B(x,y) are of the same degree.

The equation can be put in the form:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Such equation is called homogenous

Let $v = \frac{y}{x}$ ……..…………(2) ; y= v.x

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{………………. (3)}$$

$$F(v) = v + x\frac{dv}{dx}$$

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$

Examples:

1) y.dx + x.dy= 0 (homogenous/ same degree)
2) $y^2.dx + xy.dy = 0$ (homogenous/ same degree)
3) y.dx+ dy = 0 (not homogenous)
4) (y+1) dx + dy = 0 (not homogenous)
5) (y+ $\sin\frac{y}{x}$) dx + x.dy = 0 (not homogenous)
6) (y+ $x.\sin\frac{y}{x}$) dx + x.dy = 0 (homogenous/ same degree)
7) (x+y)dy + x.dx = 0 (homogenous/same degree)
8) x.dy + *siny*.dx = 0 (not homogenous)

EX (1) Solve the equation (x+y).dy – (x-y).dx = 0.

Solution:

the equation is homogenous ; $v = \frac{y}{x}$

$$F(v) = \frac{dy}{dx} = \frac{x-y}{x+y} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} = \frac{1-v}{1+v}$$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$; (homogenous) →→ $\frac{1-v}{1+v} = v + x\frac{dv}{dx}$

$$\frac{dx}{x} - \frac{dv}{\frac{1-v}{1+v}-v} = 0 \quad \rightarrow\rightarrow \quad \frac{dx}{x} - \frac{dv}{\frac{1-v-v-v^2}{1+v}} = 0$$

$$\int \frac{dx}{x} - \int \frac{(١+v)dv}{١-٢v-v^2} = 0 \rightarrow\rightarrow \quad \ln x + \frac{1}{2}\ln(١ - ٢v - v^2) = \ln c$$

$$\ln x^2 + \ln[1 - \frac{2y}{x} - (\frac{y}{x})^2] = \ln c`$$

$$x^2[1 - \frac{2y}{x} - (\frac{y}{x})^2\ ] = c`$$

$$x^2 - 2yx - y^2 = c`$$

EX (2) Solve the equation $(x^2 - y^2)$.dx – 2xy.dy = 0.

Solution:

the equation is homogenous ; v = $\frac{y}{x}$; $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$F(v) = \frac{dy}{dx} = -\left(\frac{x^2 - y^2}{2xy}\right) = \left[\frac{1+(\frac{Y}{X})^2}{2(\frac{Y}{X})}\right]$$

$$\frac{dx}{x} + \frac{dv}{v+\frac{1+v^2}{2v}} = 0 \quad \rightarrow\rightarrow\rightarrow \quad \frac{dx}{x} + \frac{dv}{\frac{2v^2+1+v^2}{2v}} = 0$$

$$\frac{dx}{x} + \frac{2v\,dv}{1+3v^2} = 0 \quad \rightarrow\rightarrow\rightarrow \quad \int\frac{dx}{x} + \int\frac{2v\,dv}{1+3v^2} = 0$$

$$\ln x + \frac{1}{3}\ln(1 + 3v^2) = \ln c$$

$$\ln x^3 + \ln(1 + 3v^2) = \ln c`$$

$$x^3\ (1 + 3v^2\ ) = c`$$

$$x^3\ (1 + 3\frac{y^2}{x^2}) = c`$$

$$x(x^2 + 3y^2\ ) = c`$$

LECTURE (2)

DIFFERENTIAL EQUATIONS
PART TWO

3- Exact Equation

A (x,y).dx + B (x,y).dy = 0

on the condition that: $\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$

Method of solution:

First we assume the solution is $\emptyset(x, y) = \text{constant}$

$$A = \frac{d\emptyset}{dx} \;\&\; B = \frac{d\emptyset}{dy}$$

$$\int d\emptyset = \int Adx$$

$$\emptyset = \int Adx$$

$$\frac{d\emptyset}{dy} = \frac{d}{dy}\int Adx = B$$

$$B = \frac{d\emptyset}{dy} = \frac{d}{dy}\int Adx$$

EX (1) Solve the equation: $(x^3$ **-**$3x^2\,y + 2x\,y^2)$**.dx – (**x^3 **-**$2x^2\,y + y^3)$**dy = 0**

Solution:

First we must check if the equation is exact.

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$$

$$A = x^3 - 3x^2\,y + 2x\,y^2;\; B = -(x^3 - 2x^2\,y + y^3)$$

$$\frac{\partial A}{\partial y} \text{ (with respect to x) } = -3\,x^2 + 4xy$$

$$\frac{\partial B}{\partial x} \text{ (with respect to y) } = -3\,x^2 + 4xy$$

Thus the equation is exact

$$\emptyset = \int A dx = \int (x^3 - 3x^2\, y + 2x\, y^2).\, \mathrm{dx}$$

$$\emptyset = \frac{x^4}{4} - x^3\, y + x^2\, y^2 + \mathrm{Cy} \text{ .... } (1)$$

where **C** is constant that may be a function of **y**.

$$\frac{d\emptyset}{dy} = -x^3 + 2x^2 y + \frac{\partial c}{\partial y} \quad ; \; \mathrm{B} = \frac{d\emptyset}{dy}$$

$$-(x^3 - 2x^2\, y + y^3) = -x^3 + 2x^2 y + \frac{\partial c}{\partial y} \longrightarrow\longrightarrow\longrightarrow \frac{\partial c}{\partial y} = -y^3$$

$$\mathrm{Cy} = -\frac{y^4}{4} - \mathrm{D} \text{ .... } (2)$$

Sub. Cy in the main equation (1);

$$\emptyset = \frac{x^4}{4} - x^3\, y + x^2\, y^2 - \frac{y^4}{4} - \mathrm{D}$$

EX (2) Solve the equation: $\sin x.\, \mathrm{dy} + y\cos x.\, dx = 0$

Solution:

First we must check if the equation is exact.

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$$

$$\mathrm{A} = y\cos x;\ \mathrm{B} = \sin x$$

$$\frac{\partial A}{\partial y} \text{ (with respect to x) } = \cos x$$

$$\frac{\partial B}{\partial x} \text{ (with respect to y) } = \cos x$$

Thus the equation is exact

$$\emptyset = \int A dx = \int y\cos x.\, \mathrm{dx}$$

$$\emptyset = y\sin x + \mathrm{Cy} \text{ .... } (1)$$

$$\frac{d\emptyset}{dy} = \sin x + \frac{\partial c}{\partial y}$$

$$B = \frac{d\emptyset}{dy} \; ; \; B = \sin x$$

$$\text{thus,} \quad \sin x = \sin x + \frac{\partial c}{\partial y} \quad \longrightarrow\longrightarrow\longrightarrow \quad \frac{\partial c}{\partial y} = 0$$

$$Cy = D \quad \dots\dots\dots\dots\dots\dots \quad (2)$$

$$\text{sub. in (1);} \quad \emptyset = y\sin x + D$$

4- Linear Equation

This type of equation has the general form**:**

$$\frac{dy}{dx} + P_x . y = Q_x \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

and solved by an integration factor (R), given by:

$$R = e^{\int P_x . dx} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

and the solution is:

$$R.y = \int R. Q_x . dx \quad + C \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

EX (1) Solve the equation: $\boldsymbol{x.\frac{dy}{dx} - y = x^3}$

Solution:

$$\frac{dy}{dx} - \frac{1}{x} y = x^2$$

$$Q_x = x^2 \; ; \; P_x = -\frac{1}{x}$$

$$R = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$$

$$R.y = \int R. Q_x . dx \quad + C$$

$$\frac{1}{x} y = \int \frac{1}{x} x^2 . dx \quad + C = \int x . dx \quad + C$$

$$\frac{1}{x} y = \frac{x^2}{2} + C \quad \longrightarrow\longrightarrow\longrightarrow \quad y = \frac{x^3}{3} + x\,C$$

EX (2) Solve the equation: $\boldsymbol{x}.\frac{dy}{dx} + 3\,y = \frac{\sin x}{x^2}$

Solution:

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3} \longrightarrow Q_x = \frac{\sin x}{x^3};\ P_x = \frac{3}{x}$$

$$R = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$R.y = \int R.Q_x.dx + C$$

$$x^3.y = \int x^3.\frac{\sin x}{x^3}dx + C = \int \sin x\,.dx + C$$

$$x^3.y = -\cos x + C$$

$$y = \frac{-\cos x}{x^3} + \frac{c}{x^3}$$

5- Bernoulli's Equation

This type of equations has a general form:

$$\frac{dy}{dx} + P_x.y = Q_x\,y^n\,;\ [n > 1]$$

The solution starts by putting the equation as:

$$\boldsymbol{y}^{-n}\left(\frac{dy}{dx}\right) + P_x.\,\boldsymbol{y}^{1-n} = Q_x \;\ldots\ldots\ldots\ldots\; (1)$$

assume; $\quad \mathbf{y}^{1-n} = w$

Differentiate with respect to $\boldsymbol{x}$

$$(1-n)\,y^{-n}\left(\frac{dy}{dx}\right) = \frac{dw}{dx}$$

$$\boldsymbol{y}^{-n}\left(\frac{dy}{dx}\right) = \frac{1}{(1-n)}\frac{dw}{dx}$$

sub. in (1) ; $\quad \frac{1}{(1-n)}\frac{dw}{dx} + P_x.w = Q_x$

$$\frac{dw}{dx} + (1-n)\,P_x.w = (1-n)\,Q_x \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2) \qquad \text{(Linear Equation)}$$

EX (1) Solve the equation: $\mathbf{y\,(6y^2 - x - 1)dx + 2x.\,dy = 0}$

Solution:

$$\frac{dy}{dx} + \frac{y\,(6y^2-x-1)}{\mathbf{2x}} = 0$$

$$\frac{dy}{dx} + \frac{-\,(x+1)}{\mathbf{2x}}\,y + \frac{6y^3}{2x} = 0$$

$$\frac{dy}{dx} - \frac{x+1}{\mathbf{2x}}\,y + \frac{3}{x}\,y^3 = 0$$

$$\frac{dy}{dx} - \frac{x+1}{\mathbf{2x}}\,y = -\,\frac{3}{x}\,y^3$$

$$y^{-3}\,\frac{dy}{dx} - \frac{\mathbf{x+1}}{\mathbf{2x}}\,\boldsymbol{y}^{-2} = -\,\frac{3}{x}$$

$$\frac{\boldsymbol{dw}}{\boldsymbol{dx}} + \textbf{(1-n)}\,\boldsymbol{P_x.\,w} = \text{(1-n) } Q_x \text{ (main equation)}$$

$$\mathbf{w} = \boldsymbol{y}^{-2}$$

$$\frac{\boldsymbol{dw}}{\boldsymbol{dx}} = \mathbf{-2}\,\boldsymbol{y}^{-3}\,\frac{\boldsymbol{dy}}{\boldsymbol{dx}}$$

Sub. in the main equation:

$$-\frac{1}{2}\frac{dw}{dx} - \frac{x+1}{2x}\,w = -\,\frac{3}{x}$$

$$\frac{dw}{dx} + \frac{x+1}{x}\,w = \frac{6}{x} \quad \text{Linear Equation}$$

Solving as linear equation;

$$\text{R} = e^{\int P_x.\text{dx}} = e^{\int \frac{x+1}{x}.\text{dx}} = e^{\int \text{dx} + \int \frac{dx}{x}}$$

$$= e^{x + \ln x} = e^x.\, e^{\ln x} = xe^x$$

$$\text{R} = xe^x$$

$$\text{R}.y = \int R.\,Q_x.\,dx \quad + \text{C}$$

$$\text{w} = y^{-2};\ Q_x = \frac{6}{x}$$

$$xe^x.\,w = \int xe^x.\frac{6}{x}\,.\,dx \quad + \text{C}$$

$$xe^{x}.w = 6\int e^{x}.\ dx \quad + C$$

$$xe^{x}.w = 6\ e^{x} + C$$

$$xe^{x}.y^{-2} = 6\ e^{x} + C$$

EX (2) Solve the equation: $\boldsymbol{6y^2dx - x(2x^3 + y)dy = 0}$

Solution:

$$6y^2dx = x(2x^3 + y)dy$$

$$\frac{dx}{dy} = \frac{x(2x^3+y)}{6y^2} = \frac{(2x^4+xy)}{6y^2} = \frac{2x^4}{6y^2} + \frac{xy}{6y^2}$$

$$\frac{dx}{dy} - \frac{x}{6y} = \frac{x^4}{3y^2} \qquad \text{(Bernoulli's Equation)}$$

$$x^{-4}.\frac{dx}{dy} - \frac{x^{-3}}{6y} = \frac{1}{3y^2} \quad \ldots\ldots\ldots\ldots\ldots\ (1)$$

$$W = x^{-3} \longrightarrow\longrightarrow \frac{dw}{dy} = -3\ x^{-4}\frac{dx}{dy}$$

$$\frac{dx}{dy} = -\frac{1}{3x^{-4}}\ \frac{dw}{dy}$$

$$\text{sub. in (1)} \qquad -\frac{1}{3}\ \frac{dw}{dy} - \frac{w}{6y} = \frac{1}{3y^2}$$

$$\frac{dw}{dy} - \frac{3w}{6y} = \frac{-3}{3y^2}$$

$$\frac{dw}{dy} + \frac{w}{2y} = \frac{-1}{y^2} \qquad \text{(Linear Equation)}$$

$$R = e^{\int\frac{1}{2y}.dx} = e^{\frac{1}{2}\ln y} = y^{\frac{1}{2}}$$

$$\boldsymbol{w = x^{-3};}\ Q_y = \frac{-1}{y^2}$$

$$y^{\frac{1}{2}}\ .\ w = \int y^{\frac{1}{2}}.\frac{-1}{y^2}.dy \quad + C = -\frac{y^{-\frac{1}{2}}}{-\frac{1}{2}} + C = 2y^{\frac{-1}{2}} + C$$

$$y^{\frac{1}{2}}\ .\ x^{-3} = 2y^{\frac{-1}{2}} + C$$