3.Arc length in polar coordinate

The length of arc in Cartesian coordinates when y=f(x) is∴

$$l = \int_a^b |v|\, dt$$

$$ds = \sqrt{dx^2 + dy^2} \rightarrow s = \int \sqrt{dx^2 + dy^2}\, ds$$

But in polar coordinate where we have

$$y = \text{r}\sin\theta \;\; .x = \text{r}\cos\theta \;.\;\; r = f(\theta)$$

$$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta}$$

$$\frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}$$

$$\frac{\text{ds}}{\text{d}\theta} = \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}$$

$$\frac{\text{ds}}{\text{d}\theta} = \sqrt{(-r\sin\theta + \cos\theta\frac{dr}{d\theta})^2 + (r\cos\theta + \sin\theta\frac{dr}{d\theta})^2}$$

$$\frac{\text{ds}}{\text{d}\theta} = \sqrt{r^2 + (\frac{dr}{d\theta})^2} \rightarrow ds = \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$$

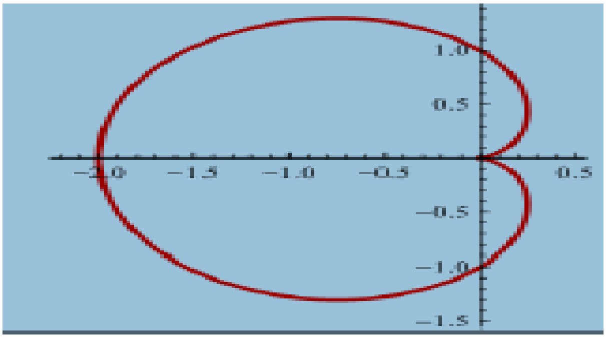
$$S = \int_{\theta 1}^{\theta 2} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$$

Example 1: Find length of the cardioid $r = a(1 - \cos\theta)$

from $\theta = 0. \theta = 2\pi$

Solution /

$$\frac{dr}{d\theta} = a\sin\theta. \quad S = \int_0^{2\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$$



$$= \int_0^{2\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2(1 - 2\cos\theta + \cos^2\theta) + a^2 \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} a\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} a\sqrt{(2 - 2\cos\theta}\, d\theta = \int_0^{2\pi} \sqrt{2}a\sqrt{(1 - \cos\theta}\, d\theta$$

$$[\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \leftrightarrow \sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)]$$

$$= \int_0^{2\pi} \sqrt{2}a\sqrt{2\sin^2\frac{\theta}{2}}\, d\theta = \int_0^{2\pi} \sqrt{2}a\sqrt{2}\sin\frac{\theta}{2}\, d\theta = 8a$$

Exercise: Find length of the cardioid $r = 2(1 + \cos\theta)$

3.10 Function of two or more variables

If (F) is a function of two independent variables we usually call the variables X and Y and the domain of (F) are a region in the xy-plane.

Domain:

Function	Domain	Range
$W=\sqrt{y-x}$	$y \geq x$	$w \geq 0$
$W=\frac{1}{xyz}$	$x,y \& z \neq 0$	$w \neq 0$
W=sin xy	X,Y real number	$-1 \leq w \leq 1$
$w=\sqrt{x^2+y^2+z^2}$	All real number	$w \geq 0$

Limits:

We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0), and write:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$

Properties of limits of functions of two variables:

The following rules hold if L, M, k are real numbers and:

$$\lim_{(x,y)\to(x_o,y_o)} f(x,y)=L \quad \text{and} \quad \lim_{(x,y)\to(x_o,y_o)} g(x,y)=M$$

1. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y)+g(x,y))=L+M$

2. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y)-g(x,y))=L-M$

3. $\lim_{(x,y)\to(x_o,y_o)} (f(x,y).g(x,y))=L.M$

4. $\lim_{(x,y)\to(x_o,y_o)} (kf(x,y))=kL$ (any number of k)

5. $\lim_{(x,y)\to(x_o,y_o)} \dfrac{f(x,y)}{g(x,y)}=\dfrac{L}{M}$ $M\neq 0$

Example:

$$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y3}=\frac{0-(0)(1)+3}{(0)^2(1)+5(0)(1)-(1)^3}=\frac{3}{0+0-1}=-3$$

3.11 Partial derivatives

$f(x,y)$ (x_o,y_o)

$$\left.\frac{\partial f}{\partial x}\right|_{(x_o,y_o)}=\lim_{h\to 0}\frac{f(x_o+h,y_o)-f(x_o,y_o)}{h}$$

$f(x,y)$ (x_o, y_o)

$$\left.\frac{\partial f}{\partial y}\right|_{(x_o,y_o)} = \frac{d}{dy} f(x_o, y)\Big|_{y=y_o} = \lim_{h\to 0}\frac{f(x_o, y_o+h)-f(x_o,y_o)}{h}$$

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

Example 1: Find $\frac{\partial f}{\partial y}.\ \frac{\partial f}{\partial x}\ for\ f(x.y) = x^2 + 2xy + y^2 + 5$

Solution//

$$\frac{\partial f}{\partial y} = 0 + 2x + 2y + 0$$

$$\frac{\partial f}{\partial x} = 2x + 2y + 0 + 0$$

Example 2: Find $\frac{\partial f}{\partial y}.\ \frac{\partial f}{\partial x}\ for\ f(x.y) = y\sin xy$

Solution//

$$\frac{\partial f}{\partial y} = y\cos xy * x + \sin xy * 1 = xy\cos xy + \sin xy$$

$$\frac{\partial f}{\partial x} = y \cos xy * y = y^2 \cos xy$$

Example **3: Find** $\frac{\partial f}{\partial z}$ $\boldsymbol{for\ f(x.y) = x\ sin(y + 3z)}$.

Solution //

$$\frac{\partial f}{\partial z} = 3x \cos\,(y + 3z)$$

Exercise: Find $\frac{\partial f}{\partial y}. \ \frac{\partial f}{\partial x}$ $\boldsymbol{for\ f(x.y) = x^2 + y^2}$

Exercise: Find $\frac{\partial f}{\partial y}. \ \frac{\partial f}{\partial x}$ $\boldsymbol{for\ f(x.y) = \frac{\sin xy^2}{x^2+1}}$

<u>Notes:-</u>

The higher order partial derivative:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Example: Find f_{xx} , f_{yy} , f_{yx} and f_{xy} if $f(x,y) = x\cos y + ye^x$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \boxed{\cos y + ye^x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = \boxed{-x\sin y + e^x}$$

$\therefore$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos y + ye^x) = ye^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-x\sin y + e^x) = -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-x\sin y + e^x) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\cos y + ye^x) = -\sin y + e^x$$

Exercise: Find $\frac{\partial f}{\partial y} . \frac{\partial f}{\partial x} . \frac{\partial^2 f}{\partial y^2} . \frac{\partial^2 f}{\partial x^2} . \frac{\partial^2 f}{\partial x \partial y} . \frac{\partial^2 f}{\partial x \partial y}$ ***for***

$$f(x.y) = e^{x^2 y} + xy^3$$

Exercise: Find $\frac{\partial f}{\partial y} . \frac{\partial f}{\partial x} . \frac{\partial^2 f}{\partial y^2} . \frac{\partial^2 f}{\partial x^2} . \frac{\partial^2 f}{\partial x \partial y} . \frac{\partial^2 f}{\partial x \partial y}$ ***for***

$$f(x.y) = xy + \frac{e^y}{y^2 + 2}$$