



Vectors المتجهات: عنوان المحاضرة:

Vectors

vectors in plane

* $\vec{A} = OA = ai + bj$
where

i, j are the fundamental unit vectors

* Length of $\vec{A} = |\vec{A}| = \sqrt{a^2 + b^2}$

* Unit vector = $\vec{U}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{ai + bj}{|\vec{A}|}$

$\vec{U}_A = \frac{a}{|\vec{A}|} i + \frac{b}{|\vec{A}|} j$ — ①

but, $\cos \alpha = \sin \beta = \frac{a}{|\vec{A}|}$ — ②

$\cos \beta = \sin \alpha = \frac{b}{|\vec{A}|}$ — ③

Sub. eq (2) & eq (3) in eq (1) give:-

* $\vec{U}_A = \cos \alpha i + \cos \beta j$

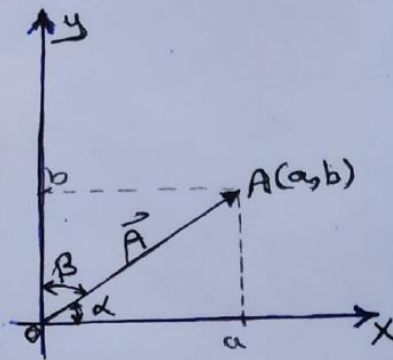
or $\vec{U}_A = \cos \alpha i + \sin \alpha j$

* $\vec{A} = \vec{U}_A \cdot |\vec{A}| = (\cos \alpha i + \sin \alpha j) \cdot |\vec{A}|$
where

a, b :- Direction numbers

α, β :- Direction angles

$\cos \alpha, \cos \beta$:- Direction Cosines



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EX.(1):- Find a vector in plane (R^2) of length (7 units) which makes angle (35°) with x-axis?

Answer :-

$$\text{Since } |\vec{A}| = 7 \quad \& \quad \alpha = 35^\circ$$

$$\vec{A} = |\vec{A}| (\cos \alpha \hat{i} + \sin \alpha \hat{j})$$

$$\therefore \vec{A} = 7 * (\cos 35^\circ \hat{i} + \sin 35^\circ \hat{j})$$

$$= \boxed{5.7 \hat{i} + 4 \hat{j}}$$

EX.(2):- Find the angle between the vector $\vec{A} = 2\hat{i} + 3\hat{j}$ and the x-axis?

Answer :-

$$|\vec{A}| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$$

$$\cos \alpha = \frac{a}{|\vec{A}|} = \frac{2}{\sqrt{13}} \Rightarrow \alpha = \cos^{-1} \left(\frac{2}{\sqrt{13}} \right)$$

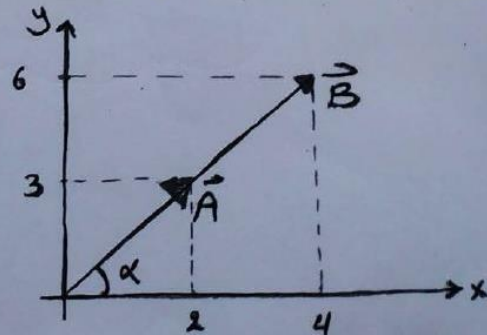
$$\alpha = \boxed{56.3^\circ}$$

Note :- Two vectors are parallel if either is proportional to an other

i.e $\vec{A} \parallel \vec{B} \iff \vec{B} = t\vec{A} \quad (\vec{u}_A = \vec{u}_B)$
where t is a scalar quantity

EX.(3):- The two vectors $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = 4\hat{i} + 6\hat{j}$ are parallel, for the reason that

$$\vec{B} = 2(2\hat{i} + 3\hat{j}) = 2\vec{A}$$





Vectors in a space

$\vec{A} = \vec{OA} = a\vec{i} + b\vec{j} + c\vec{k}$
where g_0
 $\vec{i}, \vec{j}, \vec{k}$ are the fundamental unit vectors.
 a, b, c : Direction numbers.
 α, β, γ : Direction angle
 $\cos \alpha, \cos \beta, \cos \gamma$: Direction Cosines.

* $\vec{U}_A = \frac{\vec{A}}{|\vec{A}|}$

* Length of $\vec{A} = |\vec{A}| = \sqrt{a^2 + b^2 + c^2}$

$\vec{U}_A = \frac{a\vec{i} + b\vec{j} + c\vec{k}}{|\vec{A}|}$

$\vec{U}_A = \frac{a}{|\vec{A}|}\vec{i} + \frac{b}{|\vec{A}|}\vec{j} + \frac{c}{|\vec{A}|}\vec{k}$

$\cos \alpha = \frac{a}{|\vec{A}|}$; $\cos \beta = \frac{b}{|\vec{A}|}$; $\cos \gamma = \frac{c}{|\vec{A}|}$

$\vec{U}_A = \cos \alpha \vec{i} + \cos \beta \vec{j} + \cos \gamma \vec{k}$

where

$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

and $\vec{A} = |\vec{A}| \cdot \vec{U}_A$



Ex(1) :- Find a vector in R^3 (space) of length (5 units) that makes angles (70°) with the x-axis, (85°) with the y-axis

Answer :-

$$\alpha = 70^\circ$$

$$\gamma = ?$$

$$\beta = 85^\circ$$

$$\text{Vector } \vec{A} = ?$$

$$|\vec{A}| = 5$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\cos^2 \gamma = 1 - \cos^2 70 - \cos^2 85$$

$$\cos \gamma = \sqrt{1 - \cos^2 70 - \cos^2 85} = \boxed{0.935}$$

$$\vec{A} = |\vec{A}| \cdot (\cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k})$$

$$= 5 * (\cos 70 \hat{i} + \cos 85 \hat{j} + 0.935 \hat{k})$$

$$= \boxed{5 * (1.7 \hat{i} + 0.435 \hat{j} + 0.935 \hat{k})}$$

Ex(2) :- Find the acute angle between the x-axis and the vector $\vec{A} = -4\hat{i} + 5\hat{j} + \hat{k}$?

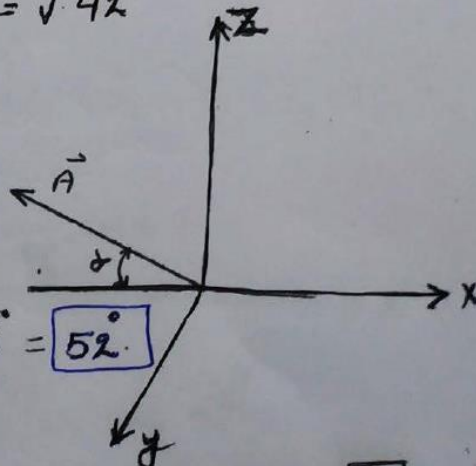
Answer :-

$$|\vec{A}| = \sqrt{(-4)^2 + (5)^2 + (1)^2} = \sqrt{42}$$

$$\cos \alpha = \frac{a}{|\vec{A}|} = \frac{-4}{\sqrt{42}} = 0.617$$

$$\alpha = \cos^{-1}(0.617) \cong 128^\circ$$

∴ The required angle is $180^\circ - 128^\circ = \boxed{52^\circ}$





Definition
1- Algebraic addition :-

$$\text{If } \vec{A} = a_1i + b_1j \text{ and } \vec{B} = a_2i + b_2j \\ \text{Then } \vec{A} + \vec{B} = (a_1 + a_2)i + (b_1 + b_2)j$$

2- Subtraction :-

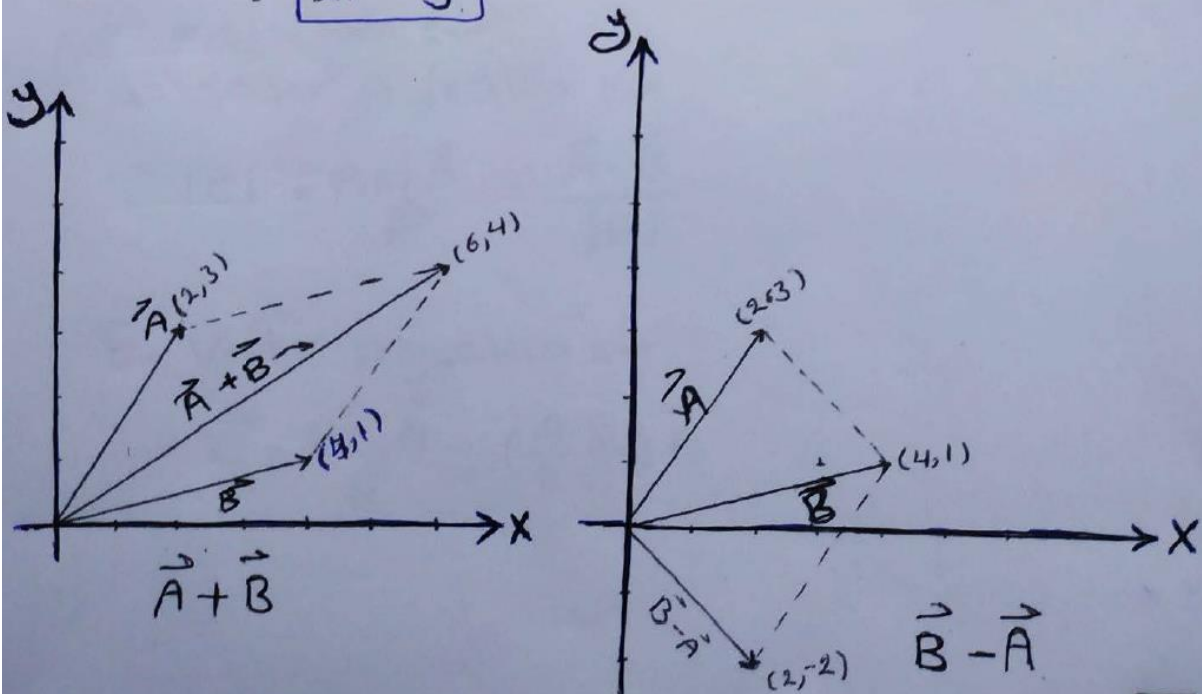
$$\vec{A} - \vec{B} = (a_1 - a_2)i + (b_1 - b_2)j$$

EX(1) :- If $\vec{A} = 2i + 3j$ and $\vec{B} = 4i + j$, then find $\vec{A} + \vec{B}$ and $\vec{B} - \vec{A}$ and sketch ?

Answer :-

$$\vec{A} + \vec{B} = (2i + 3j) + (4i + j) \\ = (2 + 4)i + (3 + 1)j \\ = \boxed{6i + 4j}$$

$$\vec{B} - \vec{A} = (4i + j) - (2i + 3j) \\ = (4 - 2)i + (1 - 3)j \\ = \boxed{2i - 2j}$$





* Scalar product (Dot-product) :-

Let $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$* \boxed{\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \cos \theta}$$

where θ is the angle between \vec{A} and \vec{B}

Properties :-

$$① \vec{A} \cdot \vec{A} = |\vec{A}|^2$$

$$② \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = a_1b_1 + a_2b_2 + a_3b_3$$

$$③ \cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| \cdot |\vec{B}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$④ \vec{A} \perp \vec{B} \iff \vec{A} \cdot \vec{B} = 0 \quad [\text{Orthogonal vectors}]$$

$$⑤ a\vec{i} + b\vec{j} \perp b\vec{i} - a\vec{j}$$

⑥ Projection :-

a- Scalar projection :-

$$|c| = \text{Proj}_{\vec{B}} \vec{A} = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$$

b- Vector projection :-

$$\vec{C} = \text{Proj}_{\vec{B}} \vec{A} = \left(\frac{\vec{A} \cdot \vec{B}}{\vec{B} \cdot \vec{B}} \right) \vec{B}$$



Ex(1) :- Find the angle θ between $\vec{A} = i - 2j - 2k$ and $\vec{B} = 6i + 3j + 2k$?

Answer :-

$$A \cdot B = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = \boxed{-4}$$

$$|A| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = \boxed{3}$$

$$|B| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = \boxed{7}$$

$$|A| \cdot |B| = 3 \cdot 7 = \boxed{21}$$

$$\therefore \cos \theta = \frac{A \cdot B}{|A| \cdot |B|} = \frac{-4}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{-4}{21}\right)$$

$$\therefore \boxed{\theta \approx 101^\circ}$$

Ex(2) :- Show that the two vectors $\vec{A} = 3i - 2j + k$ and $\vec{B} = 2j + 4k$ are orthogonal ?

Answer :-

$$A \cdot B = (3)(0) + (-2)(2) + (1)(4) \\ = 0 - 4 + 4 = 0$$

$$\therefore A \cdot B = 0$$

\therefore The two vectors \vec{A} and \vec{B} are orthogonal



Ex(3) :- Find the ^{vector} projection of $(\vec{B} = 6i + 3j + 2k)$ onto $(\vec{A} = i - 2j - 2k)$ and the scalar component of \vec{B} in the direction of A ?

Answer :- The vector projection is

$$\text{Proj}_{\vec{A}} \vec{B} = \left(\frac{\vec{A} \cdot \vec{B}}{\vec{A} \cdot \vec{A}} \right) \vec{A}$$
$$= \left(\frac{(1)(6) + (-2)(3) + (-2)(2)}{(1)^2 + (-2)^2 + (-2)^2} \right) (i - 2j - 2k)$$

$$= \frac{6 - 6 - 4}{1 + 4 + 4} (i - 2j - 2k)$$

$$= \frac{-4}{9} (i - 2j - 2k)$$

$$= \boxed{\frac{-4}{9} i + \frac{8}{9} j + \frac{8}{9} k}$$

and the scalar component is

$$\text{proj}_{\vec{A}} \vec{B} = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|}$$

$$= \frac{(1)(6) + (-2)(3) + (-2)(2)}{\sqrt{(1)^2 + (-2)^2 + (-2)^2}}$$

$$= \frac{-4}{\sqrt{9}} = \boxed{\frac{-4}{3}}$$



Vector Product (Cross Product):

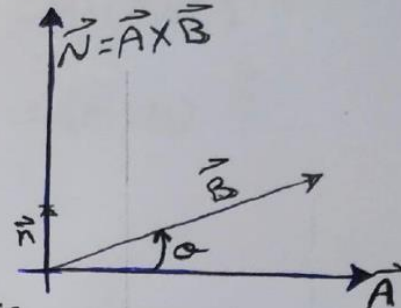
$$* \vec{N} = \vec{A} \times \vec{B} = \vec{n} |\vec{A}| |\vec{B}| \sin \theta$$

where \vec{n} is the normal unit vector normal to \vec{A} and \vec{B}

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{where}$$

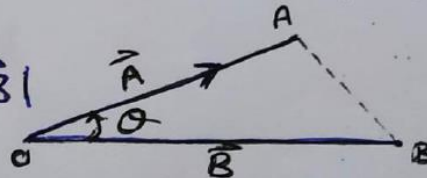
$$\vec{A} = a_1 i + a_2 j + a_3 k$$

$$\vec{B} = b_1 i + b_2 j + b_3 k$$



Properties:

- ① $\vec{A} \times \vec{A} = 0$
- ② $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- ③ $\vec{A} \parallel \vec{B} \iff \vec{A} \times \vec{B} = 0$
- ④ Area of $\Delta OAB = \frac{1}{2} |\vec{A} \times \vec{B}|$



EX(1):- Find $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ if $(\vec{A} = 2i + j + k)$
and $(\vec{B} = -4i + 3j + k)$?

Answers:

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= [(1 \times 1) - (3 \times 1)] i - [(2 \times 1) - (-4 \times 1)] j + [(2 \times 3) - (-4 \times 1)] k$$

$$\vec{A} \times \vec{B} = -2i - 6j + 10k \quad \text{but} \quad \vec{B} \times \vec{A} = -\vec{A} \times \vec{B} = 2i + 6j - 10k \quad \boxed{9}$$



Triple product :-

① Scalar triple product $(\vec{A}, \vec{B}, \vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C})$
 $= (\vec{A} \times \vec{B}) \cdot \vec{C}$

Note :- Volume of box is

$$V = |\vec{A} \cdot \vec{B} \times \vec{C}|$$

Note :- Volume of pyramid (Tetrahedron)

$$V = \frac{1}{6} |\vec{A} \cdot \vec{B} \times \vec{C}|$$

② Vector triple product

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

Note :- $i \cdot i = j \cdot j = k \cdot k = 1$

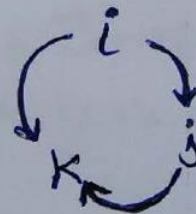
$$i \times i = j \times j = k \times k = 0$$

$$i \cdot j = j \cdot k = k \cdot i = 0$$

$$i \times j = k$$

$$j \times k = i$$

$$k \times i = j$$





H.w No.1

1) Find the length and direction of each vector and the angle it makes with the positive X-axis

(a) $i + j$

(b) $\sqrt{3}i + j$

(c) $5i + 12j$

2) Find a unit vector in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

3) Find a vector 6 units long in the direction of $\vec{A} = 2i + 2j - k$.

4) Find the length and direction of $A \times B$ and $B \times A$

(a) $\vec{A} = 2i - 2j - k$ $\vec{B} = i + j + k$

(b) $\vec{A} = 2i$ $\vec{B} = -3j$

5) Find the area of the triangle whose vertices are $A(1, -1, 0)$; $B(2, 1, -1)$ and $C(-1, 1, 2)$

6) If $\vec{A} = 2i - j$ and $\vec{B} = i + 3j - 2k$, find $A \times B$ then calculate $(A \times B) \cdot A$

7) Let $\vec{A} = 5i - j + k$; $\vec{B} = j - 5k$ and $\vec{C} = -15i + 3j - 3k$
which pairs of vector are (a) perpendicular
(b) parallel ?

----- " Vectors المتجهات " محاضرة نهاية -----