The channel capacity of the given information transmission system is defined as the maximum value of the average mutual information (I) over the channel:

or $C = I_{max}$.

Since I is function of both the channel and the source probabilities in general, then to maximize I it is required to fix the channel and then derive I with respect to source probabilities to find the probability that make I_{max}.

We shall consider just the case of symmetrical channels and for the case N=M. In this case by using the following equation:

$I = H(y) - H(y|x)$

and taking the maximization of both sides for the above equation gives:

$C = I_{max} = H_{max}(y) - H(y|x)$ or $C = \text{Log}M - H(y|x)$

In general, the capacity of symmetric channel is:

$$C = Log\ M + \sum_y P(y_j|x_i)\ LogP(y_j|x_i)$$

and for BSC the capacity is: $C = Log\ 2 + [p_e Log\ p_e + (1 - p_e).Log\ (1 - p_e)]$

or $C = 1 + [p_e Log\ p_e + (1 - p_e).Log\ (1 - p_e)]$ $\quad$ *in bits/symbol*

Q-For the case of BSC with $p_e = 0.1$, check that the capacity is C=0.531 bits/symbol

Q-Find the capacity of the ternary symmetric channel (TSC) given by:

$$P(y_j|x_i) = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.8 & 0.2 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

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Note: When calculating the channel capacity there is no need to know the source or the receiver probabilities. We just need the conditional channel matrix.

Channel Efficiency and Redundancy:

Channel Efficiency = $\eta_{ch} = \frac{I}{C}.100\%$

Channel Redundancy = $R_{ch} = \frac{C-I}{C}.100\%$

Example-6

For the following channel: $P(y_j|x_i) = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 0.1 & 0 & 0.9 \end{bmatrix}$

a) Is the channel symmetric? Why?
b) If the three source symbols probabilities are related by: $p(x_1) = p(x_2) = 2.p(x_3)$, find source probabilities, all entropies, and average mutual information.
c) Find the channel capacity, channel efficiency and redundancy.

Solution:

a) The channel is symmetric (or TSC), because the components of the rows of $P(y_j|x_i)$ are the same.
b) To start solving the problem to find source probabilities, we think that we have 3 unknowns and so we need 3 equations and these are given:
$p(x_1) = p(x_2)$ . . . (1)
$p(x_2) = 2.p(x_3)$ . . . .(2)
$p(x_1) + p(x_2) + p(x_3) = 1$ ……(3)
from (1) $p(x_2) = p(x_1)$, from (2) $p(x_3) = p(x_2)/2 = p(x_1)/2$.
Putting these relations in (3) will give:
$p(x_1) + p(x_1) + p(x_1)/2 = 1$ >>>>> $5. p(x_1) /2 = 1$ or $p(x_1) = 2/5 = 0.4$
Now, using (1) and (2) >>>> $p(x_1) = 0.4$, $p(x_2) = 0.4$, and $p(x_3) = 0.2$

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Now we have $p(x_i) = [\ 0.4 \quad 0.4 \quad 0.2]$

From the relation $P(x_i, y_j) = p(x_i)\, p(y_j|x_i)$ and the given matrix of $p(y_j|x_i)$:

$$P(x_i, y_j) = \begin{bmatrix} 0.36 & 0.04 & 0 \\ 0 & 0.36 & 0.04 \\ 0.02 & 0 & 0.18 \end{bmatrix}$$

Summing the column components to have $P(y_j) = [\ 0.38 \quad 0.4 \quad 0.22]$

Other probabilities are unnecessary in this example, now calculate:

$H(x) = -\sum_x P(x_i)\ LogP(x_i) = 1.5219$ Bit/Symbol

$H(y) = -\sum_y P(y_j)\ LogP(y_j) = 1.5398$ Bit/Symbol

-Since we have symmetric channel then, $H(y|x) = -\sum_y P(y_j|x_i)\ LogP(y_j|x_i)$

$H(y|x) = -[\ 0.9 \text{ Log } 0.9 + 0.1 \text{ Log } 0.1 + 0 \text{ Log} 0\ ] = 0.467$ Bits/Symbol

$I = H(Y) - H(Y|X) = 1.5398 - 0.467 = 1.0728$ Bits/Symbol

$H(X|Y) = H(X) - I = 1.5219 - 1.0728 = 0.4491$ Bits/Symbol

$H(X,Y) = H(X) + H(Y) - I = 1.5219 + 1.5398 - 1.0728 = 1.9889$ Bits/Symbol

c) Using the general expression for channel capacity for symmetric channel:

$$C = Log\ M - H(Y|X) = Log\ M + \sum_y P(y_j|x_i)\ LogP(y_j|x_i)$$

$$= Log_2\, 3 - 0.467 = 1.1179 \text{ Bits/Symbol}$$

Channel Efficiency $= \eta_{ch} = \frac{I}{C}.100\% = \frac{1.0728}{1.1179}.100\% = 95.96\ \%$

Channel Redundancy $= R_{ch} = \frac{C-I}{C}.100\% = \frac{1.1179-1.0728}{1.1179}.100\% = 4.04\ \%$

Q- If you have the choice of changing the parameters of above example WHAT YOU WILL DO?

So that we obtain maximum information (I=C=1.1179)

8-Entropy, Information, and Capacity Rates

The meaning of rate is the unit of a physical quantity per unit time. For the entropy, the average mutual information, and the capacity the rate is measured in bits per second (bits/sec) or more general bps. This is much important than the unit Bits/symbol.

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In terms of units >>> $\frac{Bit}{Symbol}\ x\ \frac{Symbol}{second} = \frac{Bit}{Second}\ or\ bps$

Let R_x be the source symbol rate, then the time of the symbol (T_x)is given by:

$T_x = \frac{1}{R_x}$ second/Symbol

Now each entropy H, I or C can be converted from Bits/Symbol unit into rate unit of bps by multiplying each of them by R_x, as follows:

$$H_r(x) = R_x.H(x)$$

$$I_r = R_x.I$$

$$C_r = R_x.C$$

Example-7

For Example-6, find $H_r(x), I_r\ and\ C_r$, if the average time interval of the source symbol is $10\ \mu.sec$.

Solution:

Since $\mathrm{T_x} = \frac{1}{\mathrm{R_x}}$ then $\mathrm{R_x} = \frac{1}{\mathrm{T_x}} = \frac{1}{10 \times 10^{-6}} = 10^5 = 100000$ symbols/sec

From the results of Example-6

H(x) =1.5219 Bits/Symbol, then $H_r(x) = R_x.H(x) = 152190$ bps

$I = 1.0728$ Bits/symbol , then $I_r = R_x.I = 107280$ bps

C=1.1179 Bits/symbol, then $C_r = R_x.C = 111790$ bps.

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9- Information, and Capacity Over Continues Channel

- Primarily Definitions

a) The Bandwidth (B): The bandwidth is the range of frequency occupied by given signal or system in Hz. It can be measured by the difference between f_{max} and f_{min} over positive side in the frequency domain.

b) Nyquist's Theorem: The maximum sample or symbol rate of signal over channel having bandwidth B is limited to 2B symbols/sec. In mathematical representation: $R \leq 2.B$ Symbols/sec (R_{max}=2B).

c) The signal-to-noise power ratio (S/N): It is the ratio of the signal power (S in Watt) to the noise power (N in Watt also) in the channel. Thus, it is a unit-less (ratio). Usually, expressed in dB where;

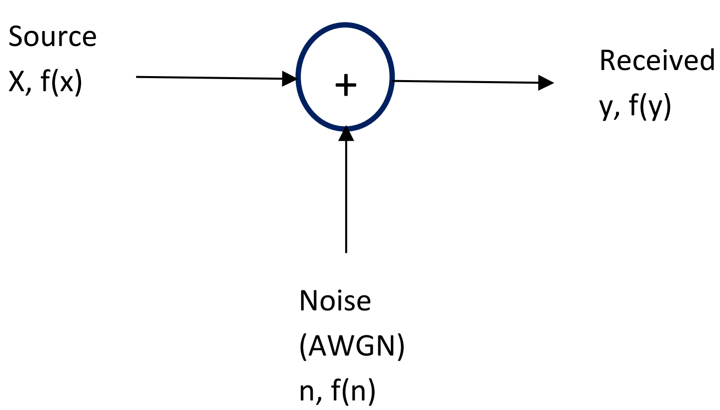
$$\left.\frac{S}{N}\right]_{dB} = 10.Log_{10}\left.\frac{S}{N}\right]_{ratio} dB$$

The inverse conversion is required also: $\left.\frac{S}{N}\right]_{ratio} = 10^{[S/N]_{dB} \div 10}$

Q- What is the definition of dBm?

- Basic Assumptions

The model in the case of continues channel is shown below:



As before, the source output is continues random variable symbols x with pdf f(x), the received symbol is y with pdf f(y), and the noise is n with pdf f(n). It is required to find an expression for C.

Assumptions / Their Reason:

1- f(x) is Gaussian (normal) /The $H_{max}(x)$ is given by Gaussian RV.
2- f(n) is also Gaussian distribution / Due to the following:
   a- Natural noise is totally random in nature like Gaussian RV.
   b- We need to test the system under worst case of noise (Gaussian).
   c- According to central limit theorem "sum of unknown independent noise sources can be modeled as Gaussian RV"
3- The mean values of the source and noise are zeros ($\bar{x} = 0$ *and* $\bar{n} = 0$)/ the DC level or the mean does not affect the information.

Since the noise is added to the signal and has Gaussian pdf, it is called Additive White Gaussian Noise (AWGN). The term white here is used to specify that the noise is present in all frequencies with the same power spectral density. So, the above model is also called AWGN channel model.

Now we shall use the above definitions and assumptions to derive the channel capacity for continues channel:

Since the source and noise are both Gaussian with zero means then:

- The signal power is $S = \overline{x^2} = \sigma_x^2$
  Then, $H(x) = Log\sqrt{2\pi e\sigma_x^2} = \frac{1}{2}Log(2\pi eS)$ (the signal entropy)
- The noise power is $N = \overline{n^2} = \sigma_n^2$
  Then, $H(n) = Log\sqrt{2\pi e\sigma_n^2} = \frac{1}{2}Log(2\pi eN)$
  or $H(y|x) = \frac{1}{2}Log(2\pi eN)$ (the noise entropy)
- Since y=x +n and both x and n are Gaussian RVs, then also y is Gaussian RV, with mean $\bar{y} = \bar{x} + \bar{n} = 0$, so, the received signal power is $(S + N) = \overline{y^2} = \sigma_y^2$, then

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$$H(y) = Log\sqrt{2\pi e\sigma_y^2} = \frac{1}{2}Log(2\pi e(S+N)) \quad \text{(the receiver entropy)}$$

Since we assume maximum source entropy over given AWGN channel, then:

$C = I_{max} = H(y) – H(y|x)$

$$= \frac{1}{2}Log\big(2\pi e(S+N)\big) - \frac{1}{2}Log(2\pi eN)$$

$$= \frac{1}{2}Log\frac{2\pi e(S+N)}{2\pi eN} = \frac{1}{2}Log\frac{S+N}{N} = \frac{1}{2}Log(1+\frac{S}{N})$$

We have, $C= \frac{1}{2}Log_2(1+\frac{S}{N})$ Bits/symbol

Using the above Nyquist's Theorem then $R_{max} = 2B$ Symbols/Sec

Thus, the capacity rate in bps is given by:

$$C_r = R_{max}.C = B\ Log_2(1+\frac{S}{N}) \quad \text{bps}$$

The equation is known as Shannon-Hartley equation for channel capacity:

$$C_r = B\ Log_2(1+\frac{S}{N}) \quad \text{bps}$$

The above equation relates the bandwidth of the channel with both the signal power and the noise power. Clearly, when B or S/N increased the capacity rate is also increased. In practice this is not always true since the noise power is also increased if the bandwidth increased, where

$$N=N_oB \quad \text{(in Watt)}$$

where B is the channel bandwidth as before and N_o is called the noise power spectral density (in Watts/Hz). It is the power of noise for each Hz of the channel bandwidth.

Example-8

a- The 4G cellular system used maximum bandwidth of 100MHz using efficient signal that provides S/N=20dB, find the maximum bit rate.

b- If the above is replaced by Huawei 5G cellular system that provides an extended bandwidth of 500 MHz, using the same signal and S/N of 20 dB, what is the percentage increase in system bit rate.

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Solution:

a- We have S/N= 20 dB, this should be converted to ratio to be used inside Shannon-Hartley Eq., thus,

$$\left.\frac{S}{N}\right]_{ratio} = 10^{[S/N]_{dB}\div 10} = 10^{20\div 10} = 10^2 = 100\ (ratio)$$

Now: $C_r = B\ Log_2(1+\frac{S}{N})$

$= 100\text{x}10^6\ Log_2(1+100) \approx 666\ Mbps$

b- Using 500 MHz bandwidth:

Now: $C_r = B\ Log_2(1+\frac{S}{N}) =$

$= 500\text{x}10^6\ Log_2(1+100) \approx 3330\ Mbps$ (or 3.3 Gbps !)

%increase in rate $= \frac{New\ Rate - Old\ Rate}{Old\ Rate}.100\%$

$= \frac{3330-666}{666}.100\%$

$= \frac{2664}{666}.100\% = 400\%$

This means, the rate (bps) in 5G is four times that of 4G.

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