AL-MUSTAQBAL UNIVERSITY COLLEGE - COMPUTER ENGINEERING DEPARTMENT.

ENGINEERING ANALAYSIS AND NUMERICAL ANALYSIS OF THE THIRD STAGE

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LECTURE ELEVEN

CONVOLUTION THEOREM

In this Section we introduce the convolution of two functions f(t), g(t) which we denote by $(f * g)(t)$. The convolution is an important construct because of the convolution theorem which allows us to find the inverse Laplace transform of a product of two transformed functions: $L^{-1}\{F(s)G(s)\} = (f * g)(t)$.

CONVOLUTION:

Let $f(t)$ and $g(t)$ be two functions of t. The **convolution** of $f(t)$ and $g(t)$ is also a function of t, denoted by $(f * g)(t)$ and is defined by the relation

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)\,dx$$

However if f and g are both **causal** functions then (strictly) $f(t), g(t)$ are written $f(t)u(t)$ and $g(t)u(t)$ respectively, so that

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)u(t-x)g(x)u(x)\,dx = \int_{0}^{t} f(t-x)g(x)\,dx$$

because of the properties of the step functions: $u(t-x) = 0$ if $x > t$ and $u(x) = 0$ if $x < 0$.

THEN THE ABOVE K-POINT IS AS FOLLOWS:

If $f(t)$ and $g(t)$ are causal functions then their convolution is defined by:

$$(f * g)(t) = \int_{0}^{t} f(t-x)g(x)\,dx$$

This is an odd looking definition but it turns out to have considerable use both in Laplace transform theory and in the modeling of linear engineering systems. The reader should note that the variable of integration is x. As far as the integration process is concerned the t-variable is (temporarily) regarded as a constant.

EXAMPLE 1:

Find the convolution of f and g if $f(t) = tu(t)$ and $g(t) = t^2u(t)$.

Solution

$$f(t-x) = (t-x)u(t-x) \quad \text{and} \quad g(x) = x^2u(x)$$

Therefore

$$\begin{aligned}(f * g)(t) = \int_0^t (t-x)x^2\,dx &= \left[\tfrac{1}{3}x^3t - \tfrac{1}{4}x^4\right]_0^t \\ &= \tfrac{1}{3}t^4 - \tfrac{1}{4}t^4 = \tfrac{1}{12}t^4\end{aligned}$$

EXAMPLE 2:

Find the convolution of $f(t) = t.u(t)$ and $g(t) = \sin t.u(t)$.

Solution

Here $f(t-x) = (t-x)u(t-x)$ and $g(x) = \sin x.u(x)$ and so

$$(f * g)(t) = \int_0^t (t-x)\sin x\,dx$$

We need to integrate by parts. We find, remembering again that t is a constant in the integration process,

$$\begin{aligned}\int_0^t (t-x)\sin x\,dx &= \Big[-(t-x)\cos x\Big]_0^t - \int_0^t (-1)(-\cos x)\,dx \\ &= [0+t] - \int_0^t \cos x\,dx \\ &= t - \Big[\sin x\Big]_0^t = t - \sin t\end{aligned}$$

so that

$$(f * g)(t) = t - \sin t \quad \text{or, equivalently, in this case} \quad (t * \sin t)(t) = t - \sin t$$

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the convolution of f(t) = t.u(t) and g(t) = sin t.u(t). In this if you are asked to find the convolution (g ∗ f)(t) in stead of (f*g)t that is, to reverse the order of f and g. Begin by writing (g ∗ f)(t) as an appropriate integral:

g(t − x) = sin(t − x).u(t − x) and f(x) = xu(x),

so (g ∗ f)(t) = $\int_0^t \sin(t - x).x\,dx$

$$\begin{aligned}(g*f)(t) &= \int_0^t \sin(t-x).x\,dx \\ &= \Big[x\cos(t-x)\Big]_0^t - \int_0^t \cos(t-x)\,dx \\ &= [t-0] + \Big[\sin(t-x)\Big]_0^t = t - \sin t\end{aligned}$$

Then the conclusion is:

(f ∗ g)(t) = (g ∗ f)(t) In other words: the convolution of f(t) with g(t) is the same as the convolution of g(t) with f(t).

To find the Laplace transform by convolution theorem.:

Ex. 3:

Obtain the Laplace transforms of f(t) = t.u(t) and g(t) = sint.u(t) and (f ∗g)(t). Begin by finding L{f(t)}, L{g(t)}: from table 1

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} \qquad \mathcal{L}\{g(t)\} = \frac{1}{s^2+1}$$

Table 1. Table of Laplace Transforms

Rule	Causal function	Laplace transform
1	$f(t)$	$F(s)$
2	$u(t)$	$\frac{1}{s}$
3	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
4	$\mathrm{e}^{-at}u(t)$	$\frac{1}{s+a}$
5	$\sin at \,.\, u(t)$	$\frac{a}{s^2+a^2}$
6	$\cos at \,.\, u(t)$	$\frac{s}{s^2+a^2}$
7	$\mathrm{e}^{-at}\sin bt \,.\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
8	$\mathrm{e}^{-at}\cos bt \, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$

Now find L{(f ∗ g)(t)}:

Answer

From Example 4 $(f*g)(t) = t - \sin t$ and so $\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{t - \sin t\} = \frac{1}{s^2} - \frac{1}{s^2+1}$

Now compare $\mathcal{L}\{f(t)\} \times \mathcal{L}\{g(t)\}$ with $\mathcal{L}\{f*g(t)\}$. What do you observe?

Answer

$$\mathcal{L}\{(f*g)(t)\} = \frac{1}{s^2} - \frac{1}{s^2+1} = \frac{1}{s^2}\left(\frac{1}{s^2+1}\right) = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

We see that the Laplace transform of the convolution of $f(t)$ and $g(t)$ is the product of their separate Laplace transforms. This, in fact, is a general result which is expressed in the statement of the **convolution theorem** which we discuss in the next subsection.

SO THE CONVOLUTION THEOREM :

Let $f(t)$ and $g(t)$ be causal functions with Laplace transforms $F(s)$ and $G(s)$ respectively, i.e. $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then it can be shown that

The Convolution Theorem

$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$ or equivalently $\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$

EXAMPLE 4:

Find the inverse transform of $\dfrac{6}{s(s^2+9)}$.

(a) Using partial fractions (b) Using the convolution theorem.

Solution

(a) $\dfrac{6}{s(s^2+9)} = \dfrac{(2/3)}{s} - \dfrac{(2/3)s}{s^2+9}$ and so

$$\mathcal{L}^{-1}\left\{\frac{6}{s(s^2+9)}\right\} = \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \tfrac{2}{3}u(t) - \tfrac{2}{3}\cos 3t.u(t)$$

(b) Let us choose $F(s) = \dfrac{2}{s}$ and $G(s) = \dfrac{3}{s^2+9}$ then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2u(t) \quad \text{and} \quad g(t) = \mathcal{L}^{-1}\{G(s)\} = \sin 3t.u(t)$$

So

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= (f*g)(t) \qquad \text{(by the convolution theorem)}\\ &= \int_0^t 2u(t-x)\sin 3x.u(x)\,dx\end{aligned}$$

Now the variable t can take any value from $-\infty$ to $+\infty$. If $t<0$ then the variable of integration, x, is negative and so $u(x)=0$. We conclude that

$$(f*g)(t) = 0 \quad \text{if} \quad t<0$$

that is, $(f*g)(t)$ is a **causal function**. Let us now consider the other possibility for t, that is the range $t \geq 0$. Now, in the range of integration $0 \leq x \leq t$ and so

$$u(t-x) = 1 \qquad u(x) = 1$$

since both $t-x$ and x are non-negative. Therefore

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= \int_0^t 2\sin 3x\,dx\\ &= \left[-\frac{2}{3}\cos 3x\right]_0^t = -\frac{2}{3}(\cos 3t - 1) \qquad t \geq 0\end{aligned}$$

Hence

$$\mathcal{L}^{-1}\{\frac{6}{s(s^2+9)}\} = -\frac{2}{3}(\cos 3t - 1)u(t)$$

which agrees with the value obtained above using the partial fraction approach.

EXAMPLE 5:

Use the convolution theorem to find the inverse transform of $H(s) = \frac{s}{(s-1)(s^2+1)}$.

Begin by choosing two functions of s, that is, $F(s)$ and $G(s)$:

Answer

Although there are many possibilities it would seem sensible to choose

$$F(s) = \frac{1}{s-1} \quad \text{and} \quad G(s) = \frac{s}{s^2+1}$$

since, by inspection, we can write down their inverse Laplace transforms:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^t u(t) \quad \text{and} \quad g(t) = \mathcal{L}^{-1}\{G(s)\} = \cos t.u(t)$$

Answer

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\{H(s)\} \\ &= \mathcal{L}^{-1}\{F(s)G(s)\} \\ &= \int_0^t f(t-x)g(x)\,dx = \int_0^t e^{t-x}u(t-x)\cos x.u(x)\,dx \end{aligned}$$

Now complete the evaluation of the integral, treating the cases t < 0 and t ≥ 0 separately:

Answer

You should find $h(t) = \frac{1}{2}(\sin t - \cos t + \mathrm{e}^t)u(t)$ since $h(t) = 0$ if $t < 0$ and

$$\begin{aligned}
h(t) &= \int_0^t \mathrm{e}^{t-x}\cos x\,dx \qquad \text{if} \quad t \geq 0 \\
&= \Big[\mathrm{e}^{t-x}\sin x\Big]_0^t - \int_0^t (-1)\mathrm{e}^{t-x}\sin x\,dx \qquad \text{(integrating by parts)} \\
&= \sin t + \Big[-\mathrm{e}^{t-x}\cos x\Big]_0^t - \int_0^t (-\mathrm{e}^{t-x})(-\cos x)\,dx \\
&= \sin t - \cos t + \mathrm{e}^t - h(t) \\
\text{or} \qquad 2h(t) &= \sin t - \cos t + \mathrm{e}^t \qquad t \geq 0
\end{aligned}$$

Finally $h(t) = \frac{1}{2}(\sin t - \cos t + \mathrm{e}^t)u(t)$

Exercises

1. Find the convolution of

   (a) $2tu(t)$ and $t^3u(t)$ (b) $\mathrm{e}^tu(t)$ and $tu(t)$ (c) $\mathrm{e}^{-2t}u(t)$ and $\mathrm{e}^{-t}u(t)$.

   In each case reverse the order to check that $(f * g)(t) = (g * f)(t)$.

2. Use the convolution theorem to determine the inverse Laplace transforms of

   (a) $\dfrac{1}{s^2(s+1)}$ (b) $\dfrac{1}{(s-1)(s-2)}$ (c) $\dfrac{1}{(s^2+1)^2}$

Answers

1. (a) $\frac{1}{10}t^5$ (b) $-t-1+\mathrm{e}^t$ (c) $\mathrm{e}^{-t}-\mathrm{e}^{-2t}$
2. (a) $(t-1+\mathrm{e}^{-t})u(t)$ (b) $(-\mathrm{e}^t+\mathrm{e}^{2t})u(t)$ (c) $\frac{1}{2}(\sin t - t\cos t)u(t)$