Complex Numbers

A complex number can be represented by an expression of the from $(a+bi)$ where

a and b : are real number

i : is a symbol with the property that $i^2=-1$ that mean ($i=\sqrt{-1}$) is called imaginary unit.

Definitions

① $Z = a+bi$ where Z : is a complex number

a : real part

b : imaginary part

Note : If $Z = a+bi$ ∴ $\bar{Z} = a-bi$

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where $\bar{Z}$: is a complex conjugate to the complex number (Z)

Ex : $Z = 3+4i$ ∴ $\bar{Z} = 3-4i$

② The absoulate value or modulus of $(a+bi)$ is defined as

$$|a+bi| = \sqrt{a^2+b^2}$$

Properties of Complex number

If $z_1 = x_1 + y_1 i = r_1 e^{i\theta_1}$

$z_2 = x_2 + y_2 i = r_2 e^{i\theta_2}$ then

① $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2) i$

② $z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2) i$

③ $k \cdot z_1 = k(x_1 + y_1 i) = k x_1 + k y_1 i$

④ $z_1 \cdot z_2 = (x_1 + y_1 i) \cdot (x_2 + y_2 i)$

$x_1 x_2 + x_1 y_2 i + x_2 y_1 i + y_1 y_2 \boxed{i^2} \to = -1$

$(x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

⑤ $z_1 \cdot \bar{z}_1 = |z_1|^2$ and $z_2 \cdot \bar{z}_2 = |z_2|^2$

⑥ $\dfrac{z_1}{z_2} = \dfrac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \dfrac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)}$

$$= \frac{(x_1 x_2 + y_1 y_2) + (x_2 y_1 - x_1 y_2) i}{x_2^2 + y_2^2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

⑦ $\left| \dfrac{z_1}{z_2} \right| = \dfrac{|z_1|}{|z_2|} = \dfrac{r_1}{r_2}$ $\quad r_2 \neq 0$

⑧ $\bar{\bar{z}}_1 = z_1$ and $\bar{\bar{z}}_2 = z_2$

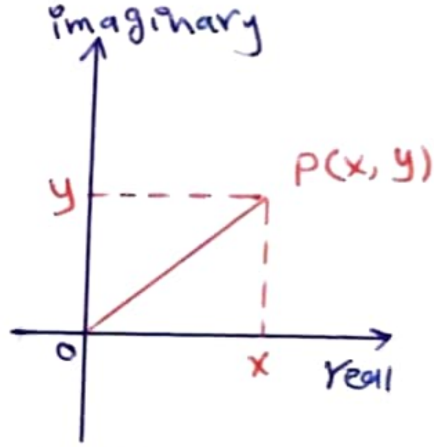
⑨ $|z_1| = \sqrt{x_1^2 + y_1^2}$ and $|z_2| = \sqrt{x_2^2 + y_2^2}$

③ If $Z_1 = a+bi$ and $Z_2 = c+di$ are equal then
$a=c$ and $b=d$

Complex number representation

There are two geometric representation of the complex number $Z = x+iy$

① The Point $P(x,y)$ in xy-Plane which is called Argand diagram or complex Plane. where x-axis represents the real axis and y-axis represent the imaginary axis.

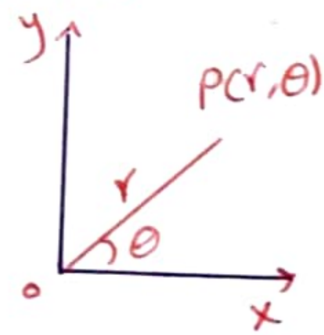


② The vector $\overrightarrow{OP}$ from the origin to $P(r,\theta)$ which is called Polar from of complex number

where:

r and θ: are called Polar coordinates.

$$Z = x+yi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$



$$r = |Z| = \sqrt{x^2+y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

where θ is the amplitude or argument angle with x-axes

$$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \text{Euler's Formula}$$

Ex let $Z_1 = 2+3i$ and $Z_2 = 4+i$

Find ① $Z_1 + Z_2$ ② $Z_1 - Z_2$

③ $Z_1 \cdot Z_2$ ④ Z_1 / Z_2

Sol ① $Z_1 + Z_2 = (2+3i) + (4+i) = \boxed{6+4i}$

② $Z_1 - Z_2 = (2+3i) - (4+i) = \boxed{-2-2i}$

③ $Z_1 \cdot Z_2 = (2+3i) * (4+i) = 8+2i+12i+3\boxed{i^2} = -1$

$= 8+14i-3 = \boxed{5+14i}$

④ $\frac{Z_1}{Z_2} = \frac{(2+3i)}{(4+i)} * \frac{(4-i)}{(4-i)} = \frac{8-2i+12i-3i^2}{(4)^2+(1)^2}$

$= \frac{8+10i+3}{17} = \frac{11+10i}{17}$

Ex Put the Complex number $(1-i\sqrt{3})$ in the Polar form.

Sol $Z = 1 - i\sqrt{3} = x+iy$

$r = |Z| = \sqrt{(1)^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-\sqrt{3}}{1} = \frac{-\pi}{3} = -60$

$Z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$= 2\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right) = 2e^{-i\frac{\pi}{3}}$

$= 2\left(\cos\left(\frac{+\pi}{3}\right) + i\sin\left(\frac{+\pi}{3}\right)\right) = 2e^{-i\frac{\pi}{3}}$

Note $\therefore$ $\ln re^{i\theta} = \ln r + i\theta$

Ex Find $\ln(-2)$

Sol $\ln(re^{i\theta}) = \ln r + i\theta$

$\ln(-2) = \ln(-2 + i0)$

$r = \sqrt{(-2)^2 + (0)^2} = 2$

$\theta = \tan^{-1} \frac{0}{-2} \Rightarrow \theta = \pi$

$\therefore \ln(-2) = \ln 2 + \pi i$

Ex Find x, y if $(3+4i) - 2(x - iy) = x + yi$

Sol $(3+4i)^2 - 2x + 2yi = x + yi$

$9 + 24i + 16i^2 - 2x + 2yi = x + yi$

$i^2 = -1 \Rightarrow 9 + 24i - 16 - 2x + 2yi = x + yi$

$-7 - 2x = x \Rightarrow -7 = x + 2x$

$3x = -7 \Rightarrow x = -7/3$

$24i + 2yi = yi \quad \div i$

$24 = y - 2y \Rightarrow -y = 24 \quad \boxed{\therefore y = -24}$