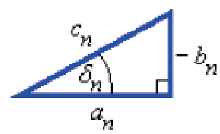
Fourier Integral and Fourier Transform

Frequency Spectrum

The Fourier series may be combined into a single cosine series. Let p be the fundamental period. If the function $f(x)$ is not periodic at all on [–L, L], then the fundamental period of the extension of $f(x)$ to the entire real line p=2L

Let the **phase** angle δ_n be such that $\tan\delta_n = -\frac{b_n}{a_n}$,

so that $\sin\delta_n = -\frac{b_n}{c_n}$ and $\cos\delta_n = +\frac{a_n}{c_n}$



where the **amplitude** is $c_n = \sqrt{a_n^2 + b_n^2}$.

Also, in the trigonometric identity $\cos A\cos B - \sin A\sin B \equiv \cos(A+B)$,

replace A by $n\omega x$ and B by δ_n. Then

$$a_n\cos(n\omega x) + b_n\sin(n\omega x) = (c_n\cos\delta_n)\cos(n\omega x) - (c_n\sin\delta_n)\sin(n\omega x)$$

$$= c_n\cos(n\omega x + \delta_n), \quad \text{where} \quad \boxed{\omega = \frac{2\pi}{p} = \frac{\pi}{L}}, \quad \boxed{c_n = \sqrt{a_n^2 + b_n^2}} \quad \text{and} \quad \boxed{\tan\delta_n = -\frac{b_n}{a_n}}$$

Therefore the phase angle or **harmonic** form of the Fourier series is

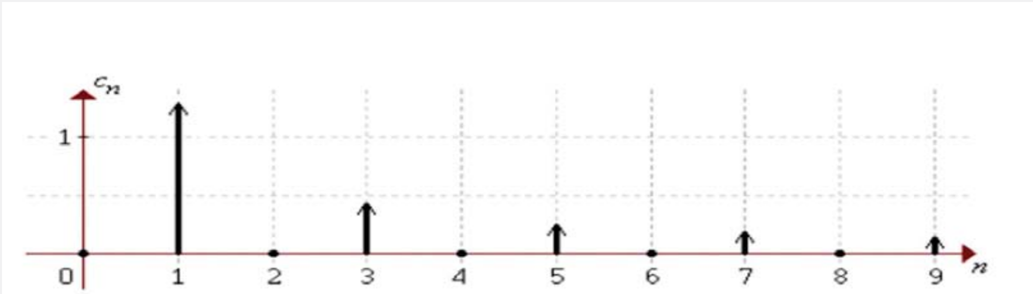
$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n\cos(n\omega x + \delta_n)}$$

Example: Plot the frequency spectrum for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

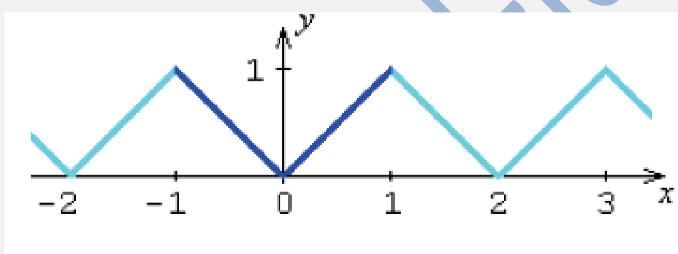
The Fourier series for the standard square wave is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\sin n\pi x\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\sin(2k-1)\pi x\right)$$

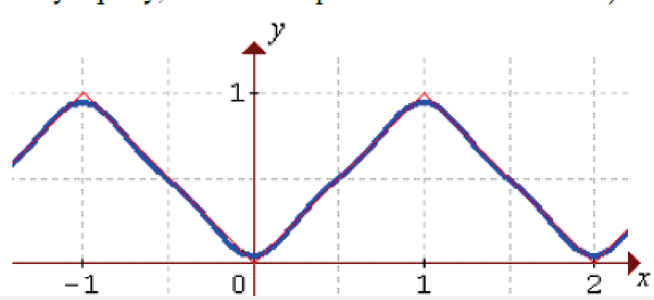


Example: Plot the frequency spectrum for the periodic extension of

$$f(x) = |x|, \quad -1 < x < 1$$



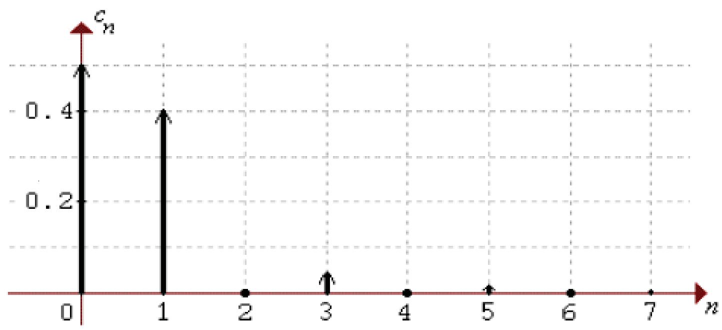
(which converges very rapidly, as this third partial sum demonstrates)



The harmonic amplitudes are

$$c_n = \left\{ \begin{array}{ll} \dfrac{1}{2} & (n=0) \\ \dfrac{2\left(1-(-1)^n\right)}{(n\pi)^2} & (n \in \mathbb{N}) \end{array} \right\} = \left\{ \begin{array}{ll} \dfrac{1}{2} & (n=0) \\ 0 & (n \text{ even}, n \geq 2) \\ \dfrac{4}{(n\pi)^2} & (n \text{ odd}) \end{array} \right.$$

The frequencies therefore diminish rapidly:



Fourier Integrals

The Fourier series may be extended from $(-L, L)$ to the entire real line.

Let $\omega_n = \dfrac{n\pi}{L} \Rightarrow \omega_n - \omega_{n-1} = \dfrac{\pi}{L} = \Delta\omega \Rightarrow \dfrac{1}{L} = \dfrac{\Delta\omega}{\pi}$

The Fourier series for $f(x)$ on $(-L, L)$ is

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \sum_{n=1}^{\infty}\left(\frac{1}{L}\left(\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt\right)\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{L}\left(\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt\right)\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$\Rightarrow f(x) = \frac{\Delta\omega}{2\pi}\int_{-L}^{L} f(t)\,dt + \sum_{n=1}^{\infty}\left(\frac{\Delta\omega}{\pi}\left(\int_{-L}^{L} f(t)\cos(\omega_n t)\,dt\right)\cos(\omega_n x) + \frac{\Delta\omega}{\pi}\left(\int_{-L}^{L} f(t)\sin(\omega_n t)\,dt\right)\sin(\omega_n x)\right)$$

Now take the limit as $L \to \infty \quad \Rightarrow \quad \Delta\omega \to 0$:

The first integral converges to some finite number, so the first term vanishes in the limit.

The summation becomes an integral over all frequencies in the limit:

$$f(x) \to 0 + \int_0^\infty \left(\frac{1}{\pi} \left(\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt \right) \cos(\omega x)\,d\omega + \frac{1}{\pi} \left(\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt \right) \sin(\omega x)\,d\omega \right)$$

Therefore the Fourier integral of $f(x)$ is

$$f(x) = \int_0^\infty \left(A_\omega \cos(\omega x) + B_\omega \sin(\omega x) \right) d\omega$$

where the Fourier integral coefficients are

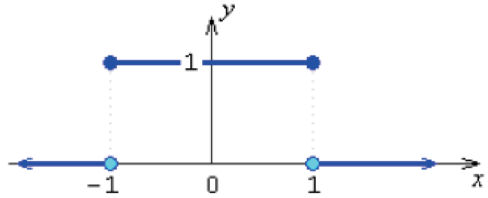
$$A_\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt \quad \text{and} \quad B_\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt$$

provided $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges.

Example: Find the Fourier integral of

$$f(x) = \begin{cases} 1 & (-1 \le x \le +1) \\ 0 & (\text{otherwise}) \end{cases}$$

From the functional form and from the graph of $f(x)$, it is obvious that $f(x)$ is piecewise smooth and that $\int_{-\infty}^{\infty}|f(x)|\,dx$ converges to the value



$$A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt = \frac{1}{\pi}\int_{-1}^{1}\cos(\omega t)\,dt = \frac{1}{\pi}\left[\frac{\sin(\omega t)}{\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}$$

The function $f(x)$ is even $\Rightarrow B_\omega = 0$ for all ω.

Therefore the Fourier integral of $f(x)$ is

$$\boxed{f(x) = \int_0^{\infty}\frac{2\sin\omega}{\pi\omega}\cos(\omega x)\,d\omega}$$

It also follows that

$$\int_0^{\infty}\frac{2\sin\omega}{\pi\omega}\cos(\omega x)\,d\omega = \begin{cases} 1 & (-1<x<1) \\ \frac{1}{2} & (x=\pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Fourier series and Fourier integrals can be used to evaluate summations and definite integrals that would otherwise be difficult or impossible to evaluate. For example, setting $x=0$ in Example 7.06.1, we find that

$$\int_0^{\infty}\frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

Complex Fourier Integrals

$$f(x) = \int_0^{\infty}\left(A_\omega\cos(\omega x) + B_\omega\sin(\omega x)\right)d\omega$$

$$= \int_0^{\infty}\left(A_\omega\frac{e^{j\omega x}+e^{-j\omega x}}{2} + B_\omega\frac{e^{j\omega x}-e^{-j\omega x}}{2j}\right)d\omega$$

$$= \int_0^{\infty}\left(\left(\frac{A_\omega - jB_\omega}{2}\right)e^{j\omega x} + \left(\frac{A_\omega + jB_\omega}{2}\right)e^{-j\omega x}\right)d\omega$$

$$= \int_0^{\infty}\left(C_\omega e^{j\omega x} + C_\omega^{*} e^{-j\omega x}\right)d\omega, \quad \text{where} \quad C_\omega = \frac{A_\omega - jB_\omega}{2}$$

$$\text{But} \quad C_\omega^* = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\cos(\omega t) + j\sin(\omega t)}{2}\,dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\cos(-\omega t) - j\sin(-\omega t)}{2}\,dt = C_{-\omega}$$

$$\text{and} \quad \int_{0}^{\infty}\left(C_{-\omega}\, e^{-j\omega x}\right)d\omega = \int_{-\infty}^{0}\left(C_{+\omega}\, e^{+j\omega x}\right)d\omega$$

By convention, the factor of $\frac{1}{2\pi}$ is extracted from the coefficients.

Therefore the complex Fourier integral of $f(t)$ is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_\omega\, e^{j\omega t}\, dt$$

where the complex Fourier integral coefficients are

$$C_\omega = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

(which is also the **Fourier transform** of f, $f(\omega) = \mathcal{F}[f(t)](\omega)$).

ω is the **frequency** of the signal $f(t)$.