Conduction through Multilayers Plane Wall

Introduction:

In heat transfer analysis, we are often interested in the rate of ***heat transfer through a medium under steady conditions*** and surface temperatures. Such problems can be solved easily without involving any differential equations by the introduction of ***thermal resistance concepts*** in an analogous manner to ***electrical circuit*** problems.

In this case, the ***thermal resistance corresponds to electrical resistance, temperature difference corresponds to voltage, and the heat transfer rate corresponds to electric current***. We start with *one-dimensional steady heat conduction* in a *multilayer* plane wall, cylinder, and sphere, and develop relations for ***thermal resistances*** in these geometries.

Consider a ***plane wall of thickness L*** and average thermal conductivity ***k***. The two surfaces of the wall are maintained at constant temperatures of T_1 and T_2. For one-dimensional steady heat conduction through the wall, we have $T(x)$. Then ***Fourier's law*** of heat conduction for the wall can be expressed as:

$$\dot{Q}_{\text{cond, wall}} = -kA\frac{dT}{dx} \qquad \text{(W)}$$

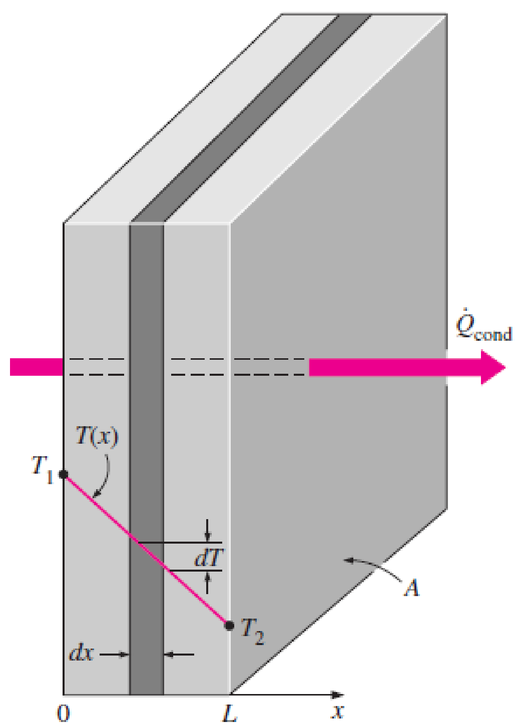
where the rate of conduction heat transfer $\dot{Q}_{\text{cond wall}}$ and the wall area ***A*** are constant. Thus we have ***dT/dx = constant***, which means that ***the temperature through the wall varies linearly with x***. That is, the temperature distribution in the wall under steady conditions is a ***straight line***. Separating the variables in the above equation and integrating from $x = 0$, where $T(0) = T_1$, to $x = L$, where $T(L) = T_2$, we get:

$$\int_{x=0}^{L} \dot{Q}_{\text{cond, wall}}\, dx = -\int_{T=T_1}^{T_2} kA\, dT$$

$$\dot{Q}_{cond,wall}\,\left[x\right]_0^L = -\,kA.\,\left[T\right]_{T1}^{T2}$$

Performing the ***integrations*** and rearranging gives

$$\dot{Q}_{cond,wall} = kA\frac{T_1 - T_2}{L} \qquad (W)$$



The Thermal Resistance Concept

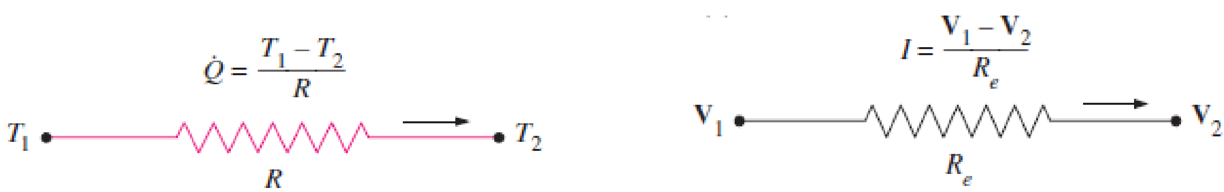
Heat conduction through a plane wall can be rearranged as

$$\dot{Q}_{cond,wall} = \frac{T_1 - T_2}{R_{wall}} \qquad (W)$$

Where: $R_{wall} = \frac{L}{kA}$ $(^oC/W)$ is the ***thermal resistance*** of the wall against heat conduction or simply the ***conduction resistance*** of the wall.

The equation above for heat flow is analogous to the relation for ***electric current flow I***, expressed as]

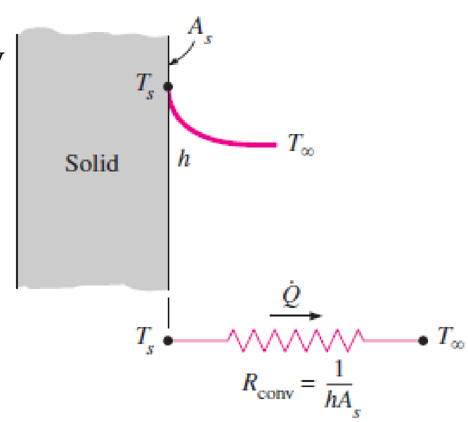
$$I = \frac{\mathbf{V}_1 - \mathbf{V}_2}{R_e}$$



(*a*) Heat flow

(*b*) Electric current flow

Consider convection heat transfer from a solid surface of area $\boldsymbol{A_s}$ and temperature $\boldsymbol{T_s}$ to a fluid whose temperature sufficiently far from the surface is $\boldsymbol{T_\infty}$, with a convection heat transfer coefficient ***h***.



Newton's law of cooling for convection heat transfer rate:

$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$ can be rearranged as:

$$\dot{Q}_{convl} = \frac{Ts - T_\infty}{R_{conv}} \qquad (W) \qquad where \quad R_{convl} = \frac{1}{hA_s} \quad (^oC/W)$$

is the ***thermal resistance*** of the surface against heat convection, or simply the ***convection resistance*** of the surface.

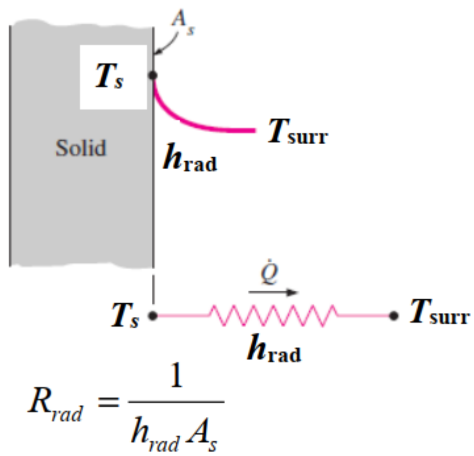
The ***rate of radiation heat transfer*** between a surface of emissivity ε and area $\boldsymbol{A_s}$ at temperature $\boldsymbol{T_s}$ and the surrounding surfaces at some average temperature $\boldsymbol{T_{surr}}$ can be expressed as:

$$\dot{Q}_{rad} = \varepsilon\sigma A_s(T_s^4 - T_{surr}^4) = \varepsilon\sigma A_s(T_s^2 - T_{surr}^2)(T_s^2 + T_{surr}^2)$$

$$\dot{Q}_{rad} = \varepsilon\sigma A_s (T_s - T_{surr})(T_s + T_{surr})(T_s^2 + T_{surr}^2)$$

$$= A_s (T_s - T_{surr}) \underbrace{\varepsilon\sigma (T_s + T_{surr})(T_s^2 + T_{surr}^2)}_{h_{rad}}$$

$$\dot{Q}_{rad} = h_{rad} A_s (T_s - T_{surr}) = \frac{T_s - T_{surr}}{R_{rad}} \qquad (W)$$



Where: $h_{rad} = \varepsilon\sigma (T_s + T_{surr})(T_s^2 + T_{surr}^2) \qquad (W/m^2 \cdot K)$

$R_{rad} = \frac{1}{h_{rad} A_s} \quad (K/W)$ is the ***thermal resistance*** of a surface against radiation, or the ***radiation resistance***, and $\boldsymbol{h_{rad}}$ is the **radiation heat transfer coefficient.**

Emissivity (ε): **0 – 1** , ***Stefan Boltzmann constant:*** **5.67 x10 $^{-8}$** (W/m^2.K)

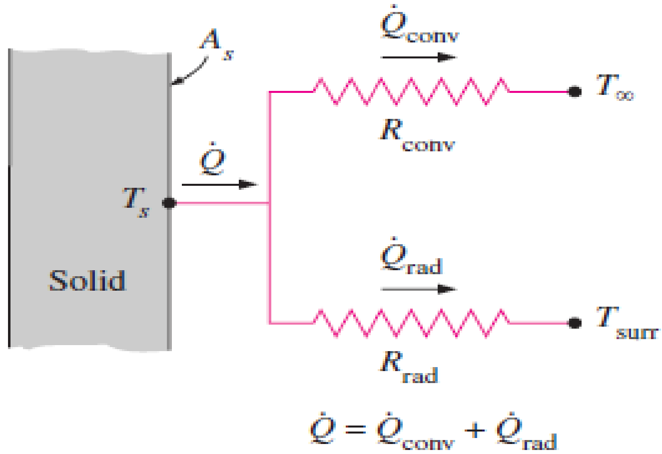
Note that both $\boldsymbol{T_s}$ and $\boldsymbol{T_{surr}}$ *must be in* **K** in the evaluation of $\boldsymbol{h_{rad}}$. The definition of the radiation heat transfer coefficient enables us to express radiation conveniently in an analogous manner to convection in terms of a temperature difference. ***But*** $\boldsymbol{h_{rad}}$ ***depends strongly on temperature while*** $\boldsymbol{h_{conv}}$ ***usually does not.***

A surface exposed to the surrounding air involves ***convection*** and ***radiation*** simultaneously, The convection and radiation resistances are ***parallel*** to each other, as shown in the Fig., and may cause some complication in the thermal resistance network.

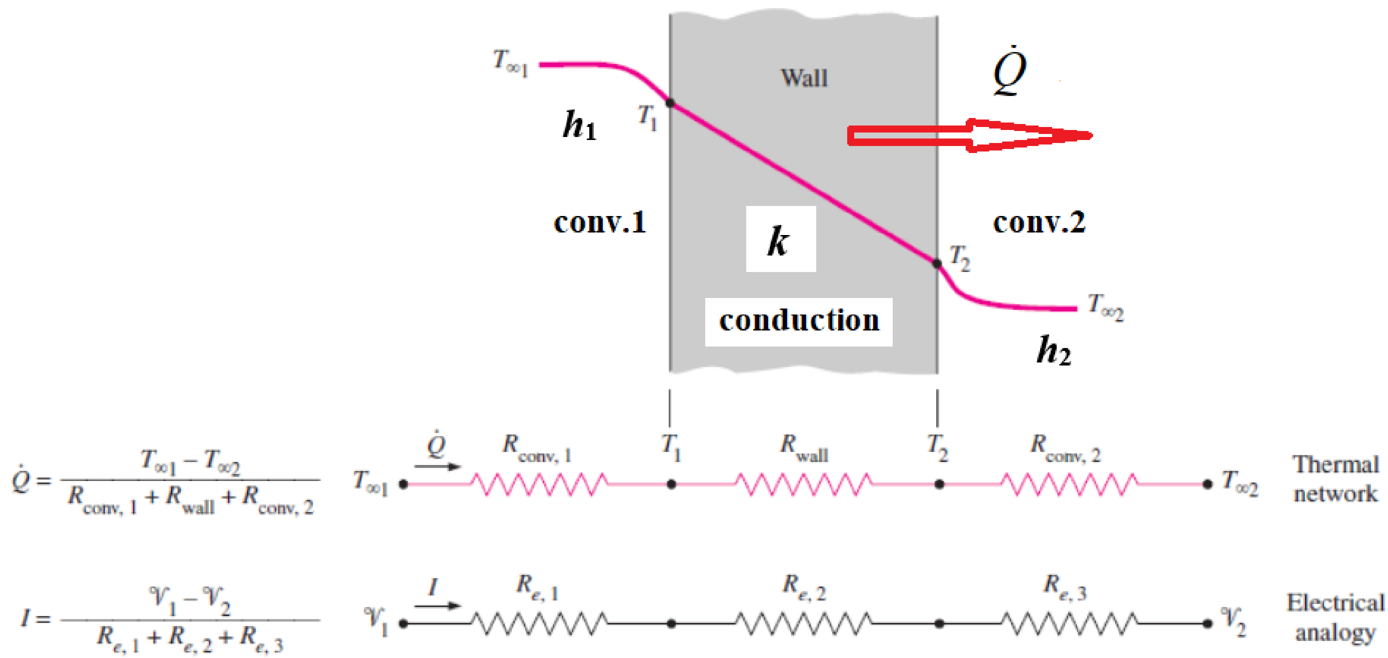
When $\boldsymbol{T_{surr} = T_\infty}$, the radiation effect can properly be accounted for by replacing $\boldsymbol{h}$ in the convection resistance relation by:

$$h_{combined} = h_{conv} + h_{rad} \qquad (W/m^2 \cdot K)$$

where $\boldsymbol{h_{combined}}$ is the ***combined heat transfer coefficient*****.** In this way, all the complications associated with radiation are avoided.



Thermal Resistance Network:



The thermal resistance network for heat transfer through a plane wall subjected to convection on both sides, and the electrical analogy.

Now consider steady one-dimensional heat flow through a plane wall of thickness $\boldsymbol{L}$, area $\boldsymbol{A}$, and thermal conductivity $\boldsymbol{k}$ that is exposed to convection on both sides to fluids at temperatures $\boldsymbol{T_{\infty 1}}$ and $\boldsymbol{T_{\infty 2}}$ with heat transfer coefficients $\boldsymbol{h_1}$ and $\boldsymbol{h_2}$, respectively, as shown in the Figure. Assuming $\boldsymbol{T_{\infty 2} < T_{\infty 1}}$, the variation of temperature will be as shown in the figure. Note that the temperature varies linearly in the wall, and asymptotically approaches $\boldsymbol{T_{\infty 1}}$ and $\boldsymbol{T_{\infty 2}}$ in the fluids as we move away from the wall.

Under steady conditions we have:

or

$$\begin{pmatrix}\text{Rate of}\\ \textit{heat convection}\\ \text{into the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\ \textit{heat conduction}\\ \text{through the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\ \textit{heat convection}\\ \text{from the wall}\end{pmatrix}$$

$$\dot{Q} = h_1 A(T_{\infty 1} - T_1) = kA\frac{T_1 - T_2}{L} = h_2 A(T_2 - T_{\infty 2})$$

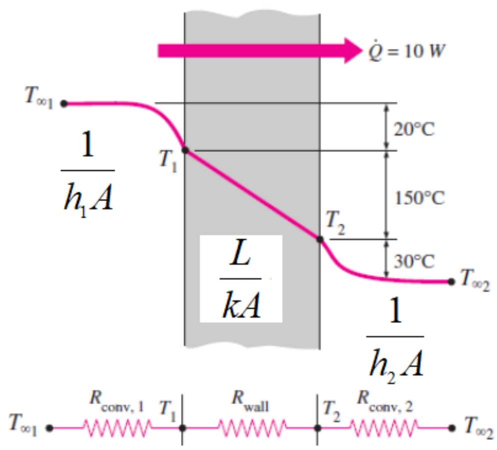
$$\dot{Q} = \frac{T_{\infty 1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/kA} = \frac{T_2 - T_{\infty 2}}{1/h_2 A} = \frac{T_{\infty 1} - T_1}{R_{\text{conv, 1}}} = \frac{T_1 - T_2}{R_{\text{wall}}} = \frac{T_2 - T_{\infty 2}}{R_{\text{conv, 2}}}$$

Adding the numerators and denominators yields,

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} \qquad (W)$$

Where $R_{total} = R_{conv,1} + R_{wall} + R_{conv,2} = \dfrac{1}{h_1 A} + \dfrac{L}{kA} + \dfrac{1}{h_2 A} \qquad (^{\circ}C/W)$

$$\dot{Q} = \frac{\Delta T}{R_{total}}$$



<u>Overall heat transfer coefficient</u> (*U*)

It is sometimes ***heat transfer through a medium*** convenient to express in an analogous manner to Newton's law of cooling as,

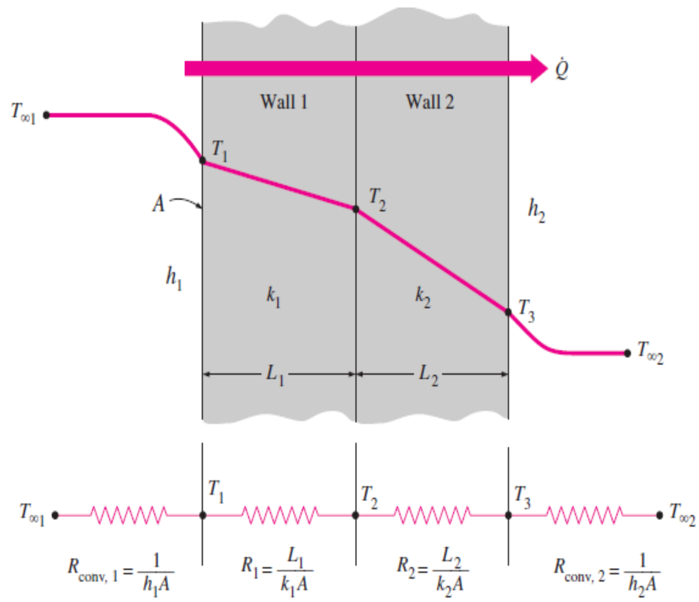
$$\dot{Q} = UA\,\Delta T \qquad (W)$$

where ***U*** is the ***overall heat transfer coefficient*** *(W/m²$\cdot$K)*.

$$UA = \frac{1}{R_{total}} \qquad (W), and, R_{total} = \frac{1}{UA}$$

Multilayer Plane Walls

In practice we often encounter plane walls that consist of ***several layers of different materials***. The thermal resistance concept can still be used to determine the rate of steady heat transfer through such ***composite walls***.



$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} \quad (W)$$

R*total** = the ***total thermal resistance, expressed as:

$$R_{total} = R_{conv,1} + R_{wall,1} + R_{wall,2} + R_{conv,2}$$

$$= \frac{1}{h_1 A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{1}{h_2 A} \qquad ({}^{o}C/W)$$

Once $\dot{Q}$ is ***known***, an unknown surface temperature T_j at any surface or interface ***j*** can be determined from:

$$\dot{Q} = \frac{T_i - T_j}{R_{\text{total}, i-j}}$$

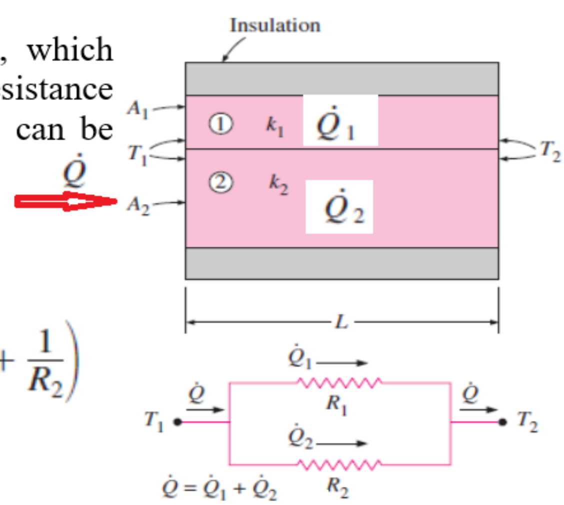
where T_i is a ***known temperature*** at location ***i*** and $R_{\text{total},i\text{-}j}$ is the total thermal resistance between locations ***i*** and ***j***.

To find T_1: $\dot{Q} = \dfrac{T_{\infty 1} - T_1}{R_{conv,1}}$ To find T_2: $\dot{Q} = \dfrac{T_{\infty 1} - T_2}{R_{conv,1} + R_1}$ To find T_3: $\dot{Q} = \dfrac{T_3 - T_{\infty 2}}{R_{conv,2}}$

Generalized Thermal Resistance Networks

The ***thermal resistance*** concept or the ***electrical analogy*** can also be used to solve steady heat transfer problems that involve parallel layers or ***combined series-parallel arrangements***.

Consider the ***composite wall*** shown in the Figure, which consists of ***two parallel layers***. The thermal resistance network, which consists of two parallel resistances, can be represented as shown in the figure.



$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} = (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Utilizing ***electrical analogy***, we get:

$$\dot{Q} = \frac{T_1 - T_2}{R_{total}} \qquad where \qquad \frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2}$$

$$\rightarrow \quad R_{total} = \frac{R_1 R_2}{R_1 + R_2}$$

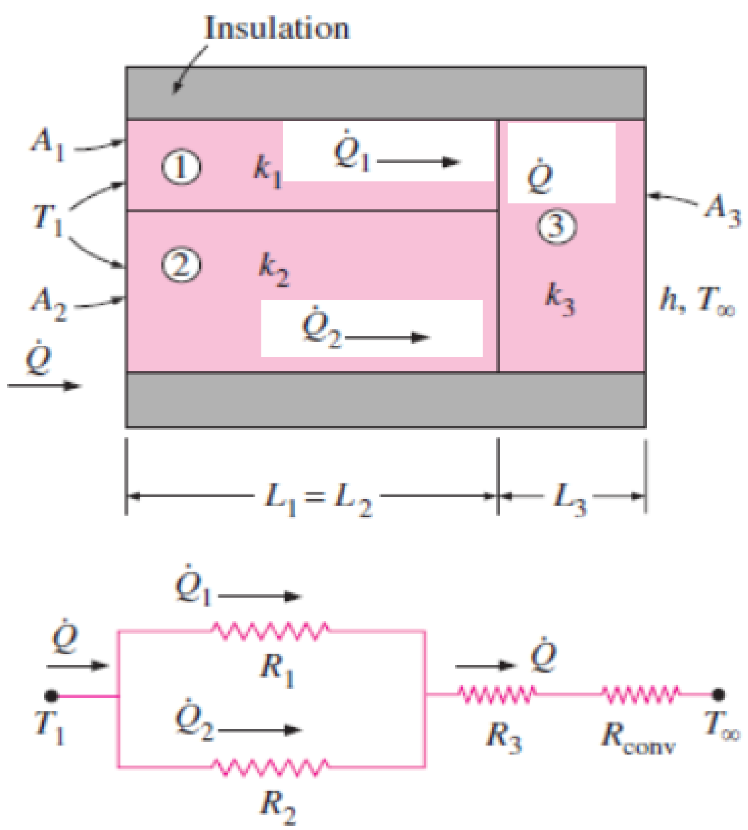
Now consider the ***combined series-parallel arrangement*** shown, the total rate of heat transfer through this composite system can again be expressed as:

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{total}}}$$

$$R_{\text{total}} = R_{12} + R_3 + R_{\text{conv}} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_{\text{conv}}$$

$$R_1 = \frac{L_1}{k_1 A_1}, \qquad R_2 = \frac{L_2}{k_2 A_2}, \qquad R_3 = \frac{L_3}{k_3 A_3}, \qquad R_{\text{conv}} = \frac{1}{hA_3}$$

Once the ***individual thermal resistances*** are evaluated, the ***total resistance and the total rate of heat transfer*** can easily be determined from the relations above.



EXAMPLE: **Heat Loss through Double-Pane Windows**

Consider a 0.8 m high and 1.5 m wide double-pane window consisting of two 4 mm thick layers of glass (k= 0.78 W/m·°C) separated by a 10 mm wide stagnant air space (***k***= 0.026 W/m·°C). ***Determine the steady rate of heat transfer through this double-pane window*** and the ***temperature of its inner surface*** for a day during which the room is maintained at 20°C while the temperature of the outdoors is -10°C. Take the convection heat transfer coefficients on the inner and outer surfaces of the window to be h_1= 10 W/m² · °C and h_2= 40 W/m²·°C, which includes the effects of radiation.

Solution:

Area: A = 0.8 x 1.5 = 1.2 m²

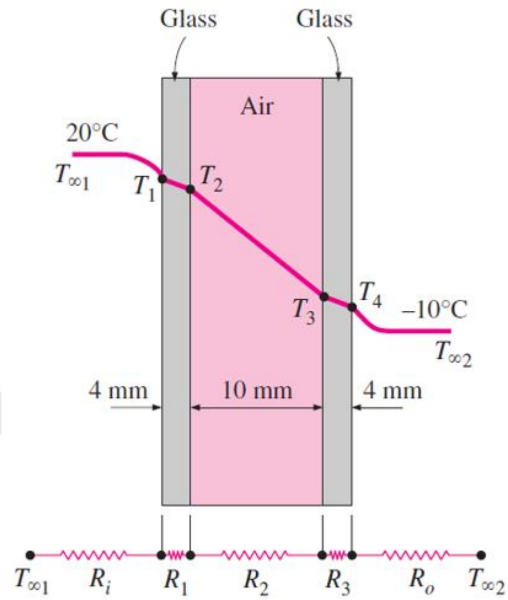
$$R_i = R_{\text{conv},1} = \frac{1}{h_1 A} = \frac{1}{(10\ \text{W/m}^2\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.08333^\circ\text{C/W}$$

$$R_1 = R_3 = R_{\text{glass}} = \frac{L_1}{k_1 A} = \frac{0.004\ \text{m}}{(0.78\ \text{W/m}\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.00427^\circ\text{C/W}$$

$$R_2 = R_{\text{air}} = \frac{L_2}{k_2 A} = \frac{0.01\ \text{m}}{(0.026\ \text{W/m}\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.3205^\circ\text{C/W}$$

$$R_o = R_{\text{conv},2} = \frac{1}{h_2 A} = \frac{1}{(40\ \text{W/m}^2\cdot{}^\circ\text{C})(1.2\ \text{m}^2)} = 0.02083^\circ\text{C/W}$$

Noting that all three resistances are in series, the total resistance is



$$\begin{aligned} R_{\text{total}} &= R_{\text{conv},1} + R_{\text{glass},1} + R_{\text{air}} + R_{\text{glass},2} + R_{\text{conv},2} \\ &= 0.08333 + 0.00427 + 0.3205 + 0.00427 + 0.02083 \\ &= 0.4332^\circ\text{C/W} \end{aligned}$$

Then the steady rate of heat transfer through the window becomes

$$\dot{Q} = \frac{T_{\infty1} - T_{\infty2}}{R_{\text{total}}} = \frac{[20 - (-10)]^\circ\text{C}}{0.4332^\circ\text{C/W}} = \mathbf{69.2\ W}$$

The inner surface temperature of the window in this case will be

$$\dot{Q} = \frac{T_{\infty1} - T_1}{R_{\text{conv},1}}$$

$$T_1 = T_{\infty1} - \dot{Q}R_{\text{conv},1} = 20^\circ\text{C} - (69.2\ \text{W})(0.08333^\circ\text{C/W}) = \mathbf{14.2^\circ C}$$