Heat transfer through the multilayered cylindrical and spherical shells

Introduction:

Steady heat transfer through a ***cylinder or sphere***, and the ***multilayered cylindrical and spherical shells*** can be handled just like ***plane walls*** by simply adding an ***additional resistance in series for each additional layer***.

Heat transfer through a pipe can be modeled as ***steady and one-dimensional***. The temperature of the pipe in this case will depend on one direction only (the radial r-direction) and can be expressed as $T = T(r)$.
In ***steady operation***, there is no ***change in the temperature of the pipe with time*** at any point. Therefore, the rate of heat transfer into the pipe must be equal to the rate of heat transfer out of it. In other words, heat transfer through the pipe must be constant, $\dot{Q}_{conduction, cyl.} =$ constant.

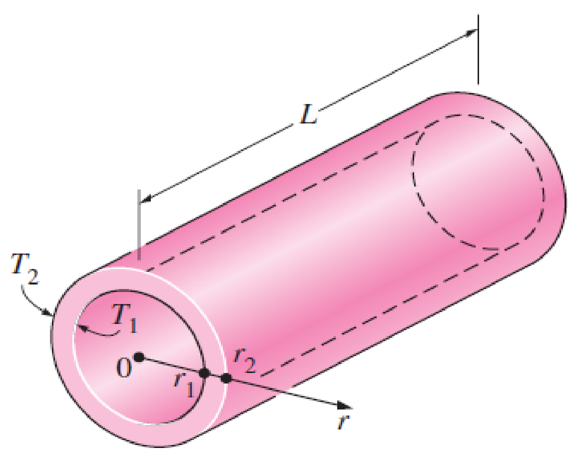
Cylindrical layer

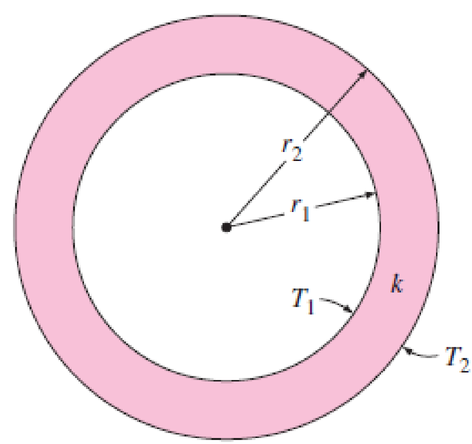
Consider a ***long cylindrical layer*** (such as a circular pipe) of **inner radius r_1, outer radius r_2, length L**, and average ***thermal conductivity k***. The two surfaces of the cylindrical layer are maintained at ***constant temperatures T_1 and T_2***.
There is ***no heat generation*** in the layer and the thermal conductivity is constant.

For ***one-dimensional heat conduction*** through the cylindrical layer, we have $T(r)$. Then ***Fourier's law*** of heat conduction for heat transfer through the ***cylindrical layer*** can be expressed as,

$$\dot{Q}_{\text{cond, cyl}} = -kA\frac{dT}{dr} \qquad \text{(W)}$$

where $A = 2\pi rL$ *or by diameter* (πDL) is the heat transfer area at location r. Note that A depends on r, and thus it ***varies in the direction of heat transfer***.





Separating the variables in the above equation and integrating from $r = r_1$, where $T(r_1) = T_1$, to $r = r_2$, where $T(r_2) = T_2$, gives:

$$\int_{r=r_1}^{r_2} \frac{\dot{Q}_{\text{cond, cyl}}}{A}\, dr = -\int_{T=T_1}^{T_2} k\, dT$$

Substituting $A = 2\pi rL$ and performing the integrations give:

$$\dot{Q}_{\text{cond, cyl}} = 2\pi Lk \frac{T_1 - T_2}{\ln(r_2/r_1)} \qquad \text{(W)}$$

$$(A = 2\pi rL)$$

$$\frac{\dot{Q}}{2\pi L} \int_{r=r_1}^{r_2} \frac{dr}{r} = -k \int_{T=T_1}^{T_2} dT$$

$$\frac{\dot{Q}}{2\pi L} (\ln r)\Big]_{r_1}^{r_2} = -k\,(T_2 - T_1)$$

$$\ln r_2 - \ln r_1 = \ln\left(\frac{r_2}{r_1}\right)$$

$$\frac{\dot{Q}}{2\pi L} \ln\left(\frac{r_2}{r_1}\right) = k\,(T_1 - T_2)$$

$$\dot{Q}_{\text{cond, cyl}} = 2\pi Lk \frac{T_1 - T_2}{\ln(r_2/r_1)}$$

since $\dot{Q}_{conduction,\,cyl.} =$ constant. This equation can be rearranged as,

$$\dot{Q}_{cond,\,cyl.} = \frac{T_1 - T_2}{\dfrac{\ln(r_2/r_1)}{2\pi Lk}} = \frac{T_1 - T_2}{R_{cyl}} \qquad (W)$$

$$\dot{Q}_{cond,\,cyl.} = \frac{T_1 - T_2}{R_{cyl}} \qquad (W)$$

where $R_{cyl.} = \dfrac{\ln(r_2/r_1)}{2\pi Lk} = \dfrac{\ln(\text{Outer radius/Inner radius})}{2\pi \text{ x (Length) x (Thermal conductivity)}}$

is the ***thermal resistance*** of the ***cylindrical layer*** against heat conduction, or simply the ***conduction resistance*** of the cylinder layer.

Spherical layer

We can repeat the analysis above for a ***spherical layer*** by taking (***surface area of sphere*** $\boldsymbol{A = 4\pi r^2}$) or ($\boldsymbol{A = \pi D^2}$) and performing the integrations, the result can be expressed as,

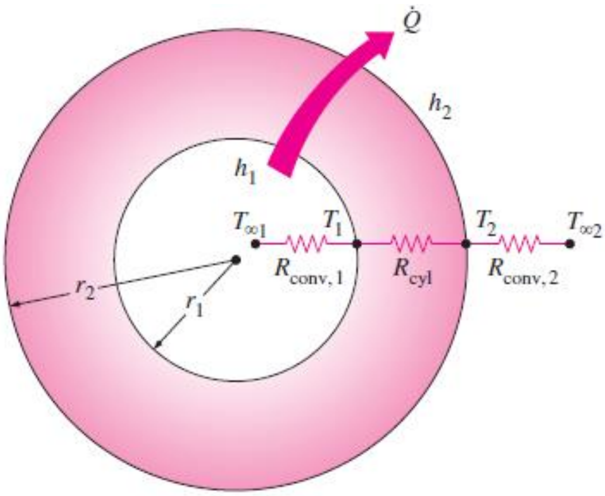
$$\dot{Q}_{cond,\,sph.} = \frac{T_1 - T_2}{R_{sph.}} \qquad (W)$$

where $R_{sph.} = \dfrac{r_2 - r_1}{4\pi\, r_1 r_2 k} = \dfrac{\text{Outer radius - Inner radius}}{4\pi \text{ x (outer radius) x (Inner radius) x (Thermal conductivity)}}$

is the ***thermal resistance*** of the ***spherical layer*** against heat conduction, or simply the ***conduction resistance*** of the spherical layer.

Now consider steady ***one-dimensional heat flow*** through a ***cylindrical or spherical layer*** that is exposed to ***convection on both sides*** to fluids at temperatures $T_{\infty 1}$ and $T_{\infty 2}$ with heat transfer coefficients $\boldsymbol{h_1}$ and $\boldsymbol{h_2}$, respectively.
The ***thermal resistance network*** in this case consists of one conduction and two convection resistances in series, just like the one for the plane wall, and the rate of heat transfer under steady conditions can be expressed as:

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} \qquad (W)$$



$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cyl}} + R_{\text{conv},2}$

Cylindrical Layer

$$R_{total} = R_{conv,1} + R_{cyl.} + R_{conv,2}$$

$$= \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{(2\pi r_2 L)h_2} \qquad (^\circ C/W)$$

for a ***cylindrical layer***, $A_1 = (2\pi r_1 L)$, $A_2 = (2\pi r_2 L)$

Spherical Layer

$$R_{total} = R_{conv,1} + R_{sph.} + R_{conv,2}$$

$$= \frac{1}{(4\pi r_1^2)h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k} + \frac{1}{(4\pi r_2^2)h_2} \qquad (^\circ C/W)$$

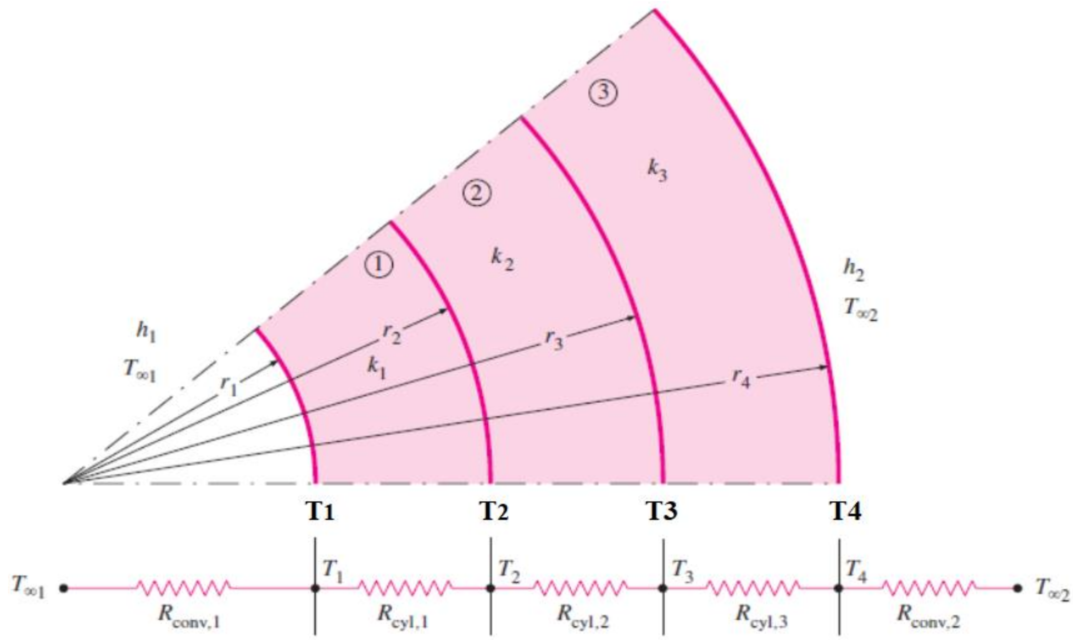
$A_1 = (4\pi r_1^2)$, $A_2 = (4\pi r_2^2)$

Note that ***A*** in the ***convection resistance*** relation $\boldsymbol{R_{conv} = 1/hA}$ is the ***surface area at which convection occurs***. It is equal to $\boldsymbol{A = 2\pi rL}$ for a ***cylindrical surface*** and $\boldsymbol{A = 4\pi r^2}$ for a ***spherical surface*** of radius ***r***. Also note that the ***thermal resistances are in series***, and thus the total thermal resistance is determined by simply adding the individual resistances, just like the ***electrical resistances*** connected in series.

Multilayered Cylinders and Spheres

Steady ***heat transfer through multilayered cylindrical or spherical shells*** can be handled just ***like multilayered plane walls*** discussed earlier by simply adding an ***additional resistance in series for each additional layer***.

For example, the steady heat transfer rate through the three-layered composite cylinder of length ***L*** with ***convection on both sides*** can be expressed as:



$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} \qquad (W)$$

<u>Multilayered Cylinder</u>

$$R_{total} = R_{conv,1} + R_{cyl.1} + R_{cyl.2} + R_{cyl.3} + R_{conv,2}$$

$$= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi L k_1} + \frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{\ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_2 A_4} \qquad ({}^{o}C/W)$$

where $A_1 = 2\pi r_1 L$ and $A_4 = 2\pi r_4 L$.

Once $\dot{Q}$ has been calculated, the interface temperature T_2 between the first and second cylindrical layers can be determined from:

$$\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{conv,\,1} + R_{cyl,\,1}} = \frac{T_{\infty 1} - T_2}{\dfrac{1}{h_1(2\pi r_1 L)} + \dfrac{\ln(r_2/r_1)}{2\pi L k_1}}$$

We could also calculate T_2 from

$$\dot{Q} = \frac{T_2 - T_{\infty 2}}{R_2 + R_3 + R_{\text{conv, 2}}} = \frac{T_2 - T_{\infty 2}}{\dfrac{\ln(r_3/r_2)}{2\pi L k_2} + \dfrac{\ln(r_4/r_3)}{2\pi L k_3} + \dfrac{1}{h_o(2\pi r_4 L)}}$$

Multilayered Sphere

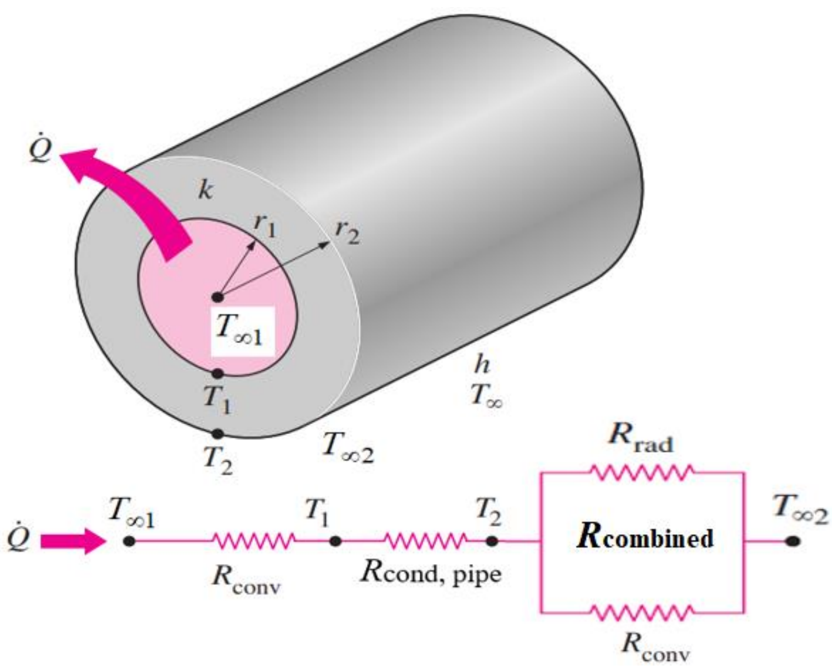
This ***equation can also be used for a three-layered spherical shell*** by replacing the thermal resistances of cylindrical layers by the ***corresponding spherical ones, (A= 4πr²)***.

$$R_{total} = R_{conv,1} + R_{sph.1} + R_{sph.2} + R_{sph.3} + R_{conv,2}$$

$$= \frac{1}{h_1 A_1} + \frac{(r_2 - r_1)/)}{4\pi r_1 r_2 k_1} + \frac{(r_3 - r_2)/)}{4\pi r_2 r_3 k_2} + \frac{(r_4 - r_3)/)}{4\pi r_3 r_4 k_3} + \frac{1}{h_2 A_4} \qquad ({}^{o}C/W)$$

Example-1:

Hot water at an average temperature of 90°C is flowing through a 15 m section of a cast iron pipe (***k*** = 52 W/m·°C) whose inner and outer diameters are 4 cm and 4.6 cm, respectively. The outer surface temperature of the pipe is 80°C with ***emissivity (ε) of 0.7*** and exposed to the cold air at 10°C in the basement, with a heat transfer coefficient of 15 W/m^2 °C. The heat transfer coefficient at the inner surface of the pipe is 120 W/m^2·°C. ***Determine the rate of heat loss from the hot water***.



Solution:

Assumptions **1** Heat transfer is steady since there is no indication of any change with time. **2** Heat transfer is one-dimensional since there is thermal symmetry about the center line and no variation in the axial direction. **3** Thermal properties are constant.

$$A_i = \pi D_i L = \pi(0.04\text{ m})(15\text{ m}) = 1.885\text{ m}^2$$
$$A_o = \pi D_o L = \pi(0.046\text{ m})(15\text{ m}) = 2.168\text{ m}^2$$

$\mathbf{r}_1 = 0.04/2 = 0.02\text{m}$, $\mathbf{r}_2 = 0.046/2 = 0.023\text{m}$

$$R_i = \frac{1}{h_i A_i} = \frac{1}{(120\text{ W/m}^2.°\text{C})(1.885\text{ m}^2)} = 0.0044\text{ °C/W}$$
$$R_{pipe} = \frac{\ln(r_2/r_1)}{2\pi k_1 L} = \frac{\ln(0.023/0.02)}{2\pi(52\text{ W/m.°C})(15\text{ m})} = 0.00003\text{ °C/W}$$

The ***radiation heat transfer coefficient*** is determined to be:

$$h_{rad} = \varepsilon\sigma(T_2{}^2 + T_{surr}{}^2)(T_2 + T_{surr})$$
$$= (0.7)(5.67\times10^{-8}\text{ W/m}^2.\text{K}^4)[(353\text{ K})^2 + (283\text{ K})^2](353+283) = 5.167\text{ W/m}^2.\text{K}$$

$\boldsymbol{T_2 = 80}$ °C $\boldsymbol{+273 = 353\ K}$ **,** $\boldsymbol{T_{surr} = 10}$ °C $\boldsymbol{+273 = 283\ K}$

Since the surrounding medium and surfaces are at the same temperature, the radiation and convection heat transfer coefficients can be added and the result can be taken as the combined heat transfer coefficient. Then,

$$h_{combined} = h_{rad} + h_{conv,2} = 5.167 + 15 = 20.167\text{ W/m}^2.°\text{C}$$
$$R_o = \frac{1}{h_{combined} A_o} = \frac{1}{(20.167\text{ W/m}^2.°\text{C})(2.168\text{ m}^2)} = 0.0229\text{ °C/W}$$
$$R_{total} = R_i + R_{pipe} + R_o = 0.0044 + 0.00003 + 0.0229 = 0.0273\text{ °C/W}$$

The rate of heat loss from the hot water pipe then becomes

$$\dot{Q} = \frac{T_{\infty1} - T_{\infty2}}{R_{total}} = \frac{(90-10)°\text{C}}{0.0273\text{ °C/W}} = \mathbf{2927\ W}$$

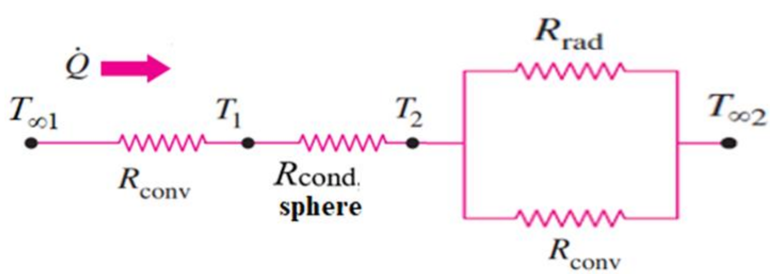
Example-2 :
A 5m internal diameter spherical tank made of 1.5cm thick stainless steel (k= 15 W/m·°C) is used to store iced water at 0°C. The tank is located in a room whose temperature is 30°C. The walls of the room are also at 30°C. The outer surface of the tank is ***black (emissivity ε =1)*** at temperature of 5°C, and ***heat transfer*** between the outer surface of the tank and the surroundings is ***by natural convection and radiation***. The convection heat transfer coefficients at the inner and the outer surfaces of the tank are 80 W/m^2 · °C and 10 W/m^2 ·°C, respectively. ***Determine (a) the rate of heat transfer to the iced water in the tank and (b) the amount of ice at 0°C that melts during a 24 hrs. period.*** The heat of fusion of water at atmospheric pressure is h_{if} = 333.7 kJ/kg.

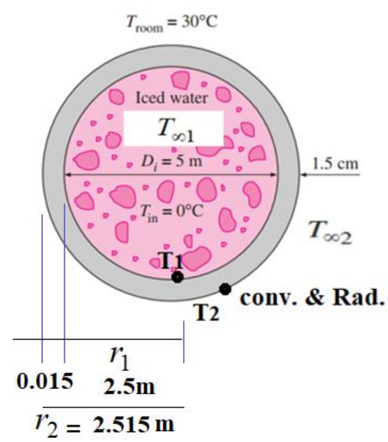
Solution:
The inner and the outer surface areas of sphere are

$$A_i = \pi D_i^{\,2} = \pi(5\,\text{m})^2 = 78.54\,\text{m}^2 \qquad A_o = \pi D_o^{\,2} = \pi(5.03\,\text{m})^2 = 79.49\,\text{m}^2$$

The radiation heat transfer coefficient can be determined from:

$$h_{rad} = \varepsilon\sigma(T_2^{\,2} + T_{surr}^{\,2})(T_2 + T_{surr})$$
$$= 1(5.67\times10^{-8}\ \text{W/m}^2.\text{K}^4)[(273+5\,\text{K})^2 + (273+30\,\text{K})^2][(273+5\,\text{K}) + (273+30\,\text{K})] = 5.570\ \text{W/m}^2.\text{K}$$





The individual thermal resistances are:

$$R_{conv,i} = \frac{1}{h_i A} = \frac{1}{(80\,\text{W/m}^2.°\text{C})(78.54\,\text{m}^2)} = 0.000159\,°\text{C/W}$$

$$R_1 = R_{sphere} = \frac{r_2 - r_1}{4\pi k r_1 r_2} = \frac{(2.515-2.5)\,\text{m}}{4\pi(15\,\text{W/m.°C})(2.515\,\text{m})(2.5\,\text{m})} = 0.000013°\text{C/W}$$

$$R_{conv,o} = \frac{1}{h_o A} = \frac{1}{(10\,\text{W/m}^2.°\text{C})(79.49\,\text{m}^2)} = 0.00126\,°\text{C/W}$$

$$R_{rad} = \frac{1}{h_{rad} A} = \frac{1}{(5.57\,\text{W/m}^2.°\text{C})(79.54\,\text{m}^2)} = 0.00226\,°\text{C/W}$$

$$\frac{1}{R_{eqv}} = \frac{1}{R_{conv,o}} + \frac{1}{R_{rad}} = \frac{1}{0.00126} + \frac{1}{0.00226} \longrightarrow R_{eqv} = 0.000809\,°\text{C/W}$$

$$R_{total} = R_{conv,i} + R_1 + R_{eqv} = 0.000159 + 0.000013 + 0.000809 = 0.000981°\text{C/W}$$

Then the steady rate of heat transfer to the iced water becomes

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} = \frac{(30-0)°\text{C}}{0.000981°\text{C/W}} = \mathbf{30{,}581\ W = 30.581\,kW}$$

$$\mathbf{= 30.581\ kJ/s}$$

(*b*) The total amount of heat transfer during a 24 hours period and the amount of ice that will melt during this period are

$$Q = \dot{Q}\Delta t = (30.581\,\text{kJ/s})(24 \times 60\text{x}60\,\text{s}) = 2.642 \times 10^6\ \text{kJ}$$

$$m_{ice} = \frac{Q}{h_{if}} = \frac{2.642 \times 10^6\ \text{kJ}}{333.7\ \text{kJ/kg}} = \mathbf{7918\,kg}$$

<u>H.W</u>

Consider a large plane wall of thickness L = 0.3 m, thermal conductivity k = 2.5 W/m · °C, and surface area A = 12 m². The left side of the wall at x = 0 is subjected to a net heat flux of q^o = 700 W/m² while the temperature at that surface is measured to be $T1$ = 80°C. Assuming constant thermal conductivity and no heat generation in the wall, **(*a*)** express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the wall, **(*b*)** obtain a relation for the variation of temperature in the wall by solving the differential equation, and **(*c*)** evaluate the temperature of the right surface of the wall at $x = L$.

***Answer:* (*c*)** ***T* = - 4°C**