Complex Fourier series

The complex exponential of Fourier series is obtained by substitution the exponential equivalent of the **Cosine** and **Sine** into the original form of Series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2} \quad \text{and} \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}}$$

$$\boxed{\cos\theta + j\sin\theta = e^{j\theta} \text{and} \ \cos\theta - j\sin\theta = e^{-j\theta}}$$

1/i=-i ; i^2=-1

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{\frac{in\pi t}{L}} + e^{\frac{-in\pi t}{L}}}{2} + b_n \frac{e^{\frac{in\pi t}{L}} - e^{\frac{-in\pi t}{L}}}{2i}\right)$$

If we define

$$C_0 = \frac{a_0}{2};\ C_n = \frac{a_n - ib_n}{2};\ C_{-n} = \frac{a_n + ib_n}{2}$$

The last series can be written

$$f(x) = \sum_{-\infty}^{\infty} \boldsymbol{C_n e^{\frac{in\pi x}{L}}}$$

$$C_0 = \frac{a_0}{2} = \frac{1}{2L}\int f(x)dx$$

$$C_n = \frac{1}{2L}\int f(x)\, \boldsymbol{e^{\frac{-in\pi x}{L}}}$$

$$C_{-n} = \frac{1}{2L}\int f(x)\, \boldsymbol{e^{\frac{in\pi x}{L}}}$$

Example: Find the complex form of Fourier series whose definition in one period

f(t)= e^{-t} -1 < t < 1

Sol:

2L=2 →L=1

$$C_n = \frac{1}{2L}\int f(x)\, e^{\frac{-in\pi x}{L}}$$

$$C_n = \frac{1}{2}\int e^{-t}\, e^{\frac{-in\pi t}{L}}$$

$$= \frac{1}{2}\left(\frac{e^{-(1+in\pi)t}}{-(1+in\pi)}\right)$$

$$\frac{e.e^{in\pi} - e^{-1}.e^{-in\pi}}{-2(1+in\pi)}$$

$$e^{i\pi} = \cos\pi + \mathrm{i}\sin\pi = -1$$

$$e^{in\pi} = (-1)^n$$

$$C_n = \frac{(-1)^n Sinh1}{(1+in\pi)} \times \frac{1-in\pi}{1-in\pi}$$

$$C_n = \frac{1 - in\pi\,(-1)^n Sinh1}{(1+n^2\pi^2)}$$

The expansion of f(t) form can be written as:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1-in\pi\,(-1)^n Sinh1\, e^{in\pi t}}{(1+n^2\pi^2)}$$

The expansion can be converted into real trigonometric form

$$C_n = \frac{a_{n-ib_n}}{2};\; C_{-n} = \frac{a_{n+ib_n}}{2}$$

$$a_n = C_n + C_{-n}$$

$$b_n = i\,(C_n - C_{-n})$$

$$a_n = \frac{(-1)^n 2\, Sinh1}{(1+n^2\pi^2)}$$

$$b_n = i\left[\frac{1-in\pi\,(-1)^n Sinh1}{(1+n^2\pi^2)} - \frac{1+in\pi\,(-1)^n Sinh1}{(1+n^2\pi^2)}\right] = \frac{2n\pi\,(-1)^n Sinh1}{1+n^2\pi^2}$$

$$C_0 = \frac{a_0}{2};\; a_0 = 2C_0 = Sinh1$$

$$f(t) = Sinh1 - 2Sinh1\left(\frac{\cos \pi t}{1+\pi^2} - \frac{\cos 2\pi t}{1+4\pi^2} + \cdots\right) - 2\pi Sinh1\left(\frac{\sin \pi t}{1+\pi^2} - \frac{2\sin 2\pi t}{1+4\pi^2}\right) + \cdots$$

H.W.

Q1) Find the complex form of Fourier series of the following functions:

1. f(t)= e^t -1 < t < 1

2. f(t)= $\begin{bmatrix} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{bmatrix}$

Applications of Fourier Series in Circuit Analysis

Effective Values and Power

The effective or rms value of the function

$$f(t) = \tfrac{1}{2}a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots + b_1 \sin \omega t + b_2 \sin 2\omega t + \cdots$$

$$F_{\text{rms}} = \sqrt{(\tfrac{1}{2}a_0)^2 + \tfrac{1}{2}a_1^2 + \tfrac{1}{2}a_2^2 + \cdots + \tfrac{1}{2}b_1^2 + \tfrac{1}{2}b_2^2 + \cdots} = \sqrt{c_0^2 + \tfrac{1}{2}c_1^2 + \tfrac{1}{2}c_2^2 + \tfrac{1}{2}c_3^3 + \cdots}$$

In general, we may write

$$v = V_0 + \sum V_n \sin(n\omega t + \phi_n) \quad \text{and} \quad i = I_0 + \sum I_n \sin(n\omega t + \psi_n)$$

With corresponding effective values of

$$V_{\text{rms}} = \sqrt{V_0^2 + \tfrac{1}{2}V_1^2 + \tfrac{1}{2}V_2^2 + \cdots} \quad \text{and} \quad I_{\text{rms}} = \sqrt{I_0^2 + \tfrac{1}{2}I_1^2 + \tfrac{1}{2}I_2^2 + \cdots}$$

The average power P follows from integration of the instantaneous power, which is given by the product of v and i:

$$p = vi = \left[V_0 + \sum V_n \sin(n\omega t + \phi_n)\right]\left[I_0 + \sum I_n \sin(n\omega t + \psi_n)\right]$$

Since v and i both have period T. **The average may therefore be calculated over one** period of the voltage wave:

$$P = \frac{1}{T}\int_0^T \left[V_0 + \sum V_n \sin(n\omega t + \phi_n)\right]\left[I_0 + \sum I_n \sin(n\omega t + \psi_n)\right] dt$$

Then the average power is

$$P = V_0 I_0 + \tfrac{1}{2} V_1 I_1 \cos\theta_1 + \tfrac{1}{2} V_2 I_2 \cos\theta_2 + \tfrac{1}{2} V_3 I_3 \cos\theta_3 + \cdots$$

Where $\theta_n = \phi_n - \psi_n$ is the angle on the equivalent impedance of the network at the angular frequency n!

$$P = \tfrac{1}{2} V_1 I_1 \cos\theta_1 = V_{\text{eff}} I_{\text{eff}} \cos\theta$$

$$P = V_0 I_0 = VI$$

$$P = P_0 + P_1 + P_2 + \cdots$$

Example: A series RL circuit in which R =5 Ω and L = 20 mH has an applied voltage as in Fig.1:

$$v = 100 + 50\sin\omega t + 25\sin 3\omega t \text{ (V), with } \omega = 500 \text{ rad/s.}$$

Find the current and the average power. Compute the equivalent impedance of the circuit at each frequency found in the voltage function. Then obtain the respective currents.

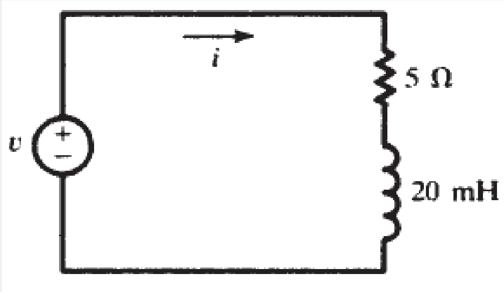

Fig.1

At $\omega = 0$, $Z_0 = R = 5\ \Omega$ and

$$I_0 = \frac{V_0}{R} = \frac{100}{5} = 20\ \text{A}$$

At $\omega = 500$ rad/s, $\mathbf{Z}_1 = 5 + j(500)(20 \times 10^{-3}) = 5 + j10 = 11.15\angle 63.4^\circ\ \Omega$ and

$$i_1 = \frac{V_{1,\max}}{Z_1}\sin(\omega t - \theta_1) = \frac{50}{11.15}\sin(\omega t - 63.4^\circ) = 4.48\sin(\omega t - 63.4^\circ) \quad \text{(A)}$$

At $3\omega = 1500$ rad/s, $\mathbf{Z}_3 = 5 + j30 = 30.4\angle 80.54^\circ\ \Omega$ and

$$i_3 = \frac{V_{3,\max}}{Z_3}\sin(3\omega t - \theta_3) = \frac{25}{30.4}\sin(3\omega t - 80.54^\circ) = 0.823\sin(3\omega t - 80.54^\circ) \quad \text{(A)}$$

The sum of the harmonic currents is the required total response; it is a Fourier series of the type (8).

$$i = 20 + 4.48\sin(\omega t - 63.4^\circ) + 0.823\sin(3\omega t - 80.54^\circ) \quad \text{(A)}$$

This current has the effective value

$$I_{\text{eff}} = \sqrt{20^2 + (4.48^2/2) + (0.823^2/2)} = \sqrt{410.6} = 20.25\ \text{A}$$

which results in a power in the 5-Ω resistor of

$$P = I_{\text{eff}}^2 R = (410.6)5 = 2053\ \text{W}$$

As a check, we compute the total average power by calculating first the power contributed by each harmonic and then adding the results.

At $\omega = 0$:	$P_0 = V_0 I_0 = 100(20) = 2000$ W
At $\omega = 500$ rad/s:	$P_1 = \frac{1}{2}V_1 I_1 \cos\theta_1 = \frac{1}{2}(50)(4.48)\cos 63.4^\circ = 50.1$ W
At $3\omega = 1500$ rad/s:	$P_3 = \frac{1}{2}V_3 I_3 \cos\theta_3 = \frac{1}{2}(25)(0.823)\cos 80.54^\circ = 1.69$ W
Then,	$P = 2000 + 50.1 + 1.69 = 2052$ W

Another Method

The Fourier series expression for the voltage across the resistor is

$$v_R = Ri = 100 + 22.4\sin(\omega t - 63.4^\circ) + 4.11\sin(3\omega t - 80.54^\circ) \quad \text{(V)}$$

and
$$V_{Reff} = \sqrt{100^2 + \frac{1}{2}(22.4)^2 + \frac{1}{2}(4.11)^2} = \sqrt{10\,259} = 101.3 \text{ V}$$

Then the power delivered by the source is $P = V_{Reff}^2/R = (10\,259)/5 = 2052$ W.

Example: Find the average power supplied to a network if the applied voltage and resulting current are

$$v = 50 + 50\sin 5\times 10^3 t + 30\sin 10^4 t + 20\sin 2\times 10^4 t \quad \text{(V)}$$
$$i = 11.2\sin(5\times 10^3 t + 63.4^\circ) + 10.6\sin(10^4 t + 45^\circ) + 8.97\sin(2\times 10^4 t + 26.6^\circ) \quad \text{(A)}$$

Sol:

The total average power is the sum of the harmonic powers:

$$P = (50)(0) + \tfrac{1}{2}(50)(11.2)\cos 63.4^\circ + \tfrac{1}{2}(30)(10.6)\cos 45^\circ + \tfrac{1}{2}(20)(8.97)\cos 26.6^\circ = 317.7 \text{ W}$$

Example: Find the trigonometric Fourier series for the half-wave-rectified sine wave shown in Fig. 2 and sketch the line spectrum

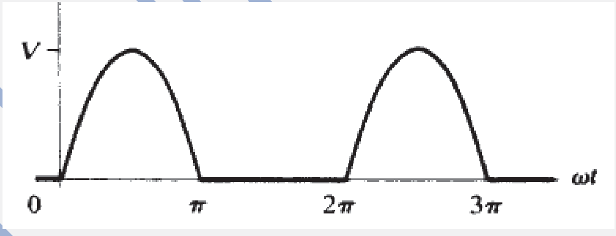

Fig.2

Sol:

The wave shows no symmetry, and we therefore expect the series to contain both sine and cosine terms. Since the average value is not obtainable by inspection, we evaluate a_0 for use in the term a_0=2.