Even and Odd Functions "Half-Range Expansions"

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval (0,L) which is half of the interval(-L,L) thus accounting for the name half range] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely,(-L,0). In such case, we have

$a_0=0$ $\quad a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$ for *half range sine series*

$b_n = 0, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$ for *half range cosine series*

Fourier Sine series:

An *odd function* is a function with the property $f(-x) = -f(x)$.

For example :

1. $f(x) = x^3$. let $x = -1$, then $(-1)^3 = -(1)^3$
2. $f(x) = \sin(x)$. let $x = -\pi/2$, then $\sin(-\pi/2) = -\sin(\pi/2)$.

Note

1. The integral of an odd function over a symmetric interval is zero.
2. Since $a_n = 0$, all the cosine functions will not appear in the Fourier series of an odd function. The Fourier series of an odd function is an infinite series of Odd functions

Let us calculate the Fourier coefficients of an odd function:

$a_0 = an = 0$

but bn $\neq 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{for half range sine series}$$

$$\mathbf{f(x)} = \sum_{n=1}^{\infty} \mathbf{bn}\sin\frac{\mathbf{n\pi x}}{\mathbf{L}}\mathbf{dx}$$

Fourier Cosine Series:

An even function is a function with the property $f(-x) = f(x)$. The sine coefficients of a Fourier series will be zero for an even function,

$f(x) = x^2$, let $x = -1$, then (-1)2 = (1)2
$f(x) = \cos(x)$, let $x = -\pi$, then $\cos(-\pi) = \cos(\pi)$.
The Fourier series of an even function is an infinite series of even functions (cosines):

Let us calculate the Fourier coefficients of an odd function:

bn =0

But

a_0=an≠ 0

$$a_0 = \frac{2}{L}\int_0^l f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{for half range cosine series}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\ \cos\frac{n\pi x}{L}\,dx$$

Example: Let us consider the function $f(x) = 1$ on $[0, \pi]$. The Fourier cosine series has coefficients

$$a_0 = \frac{2}{\pi}\int_0^{\pi} 1dx = 2$$

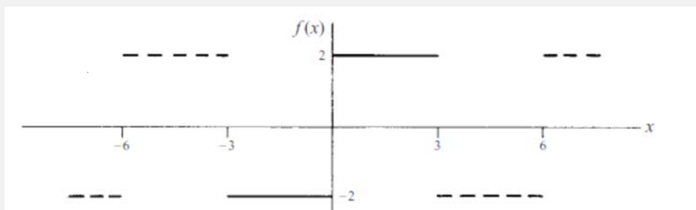
$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos(nx)dx = 0$$

Then $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = 1.$

Example: Classify each of the following functions according as they are even, odd, or neither even nor odd.

$$(a)\ f(x) = \begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases} \qquad \text{Period} = 6$$

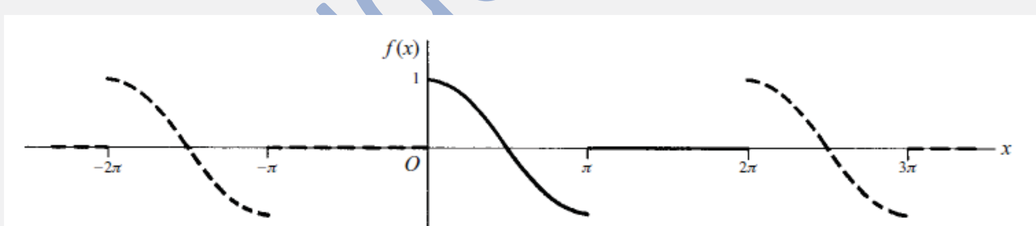
Sol: From fig.1 below it is seen that f(-x)= -f(x), so that the function is odd.



Fig(1)

$$(b)\ f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases} \qquad \text{Period} = 2\pi$$

Sol: From fig.2 below it is seen that is neither even nor odd.



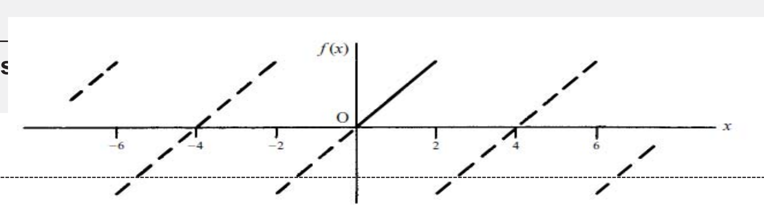
Fig(2)

Example: Expand $f(x) = x, \quad 0 < x < 2$ **, in a half range**

a) Sine series, (b) cosine series.

(a) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 3, below. This is sometimes called the odd extension of f (x) ,

Then $2L = 4, L = 2$



Thus $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$
$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then
$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$
$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

b- H.W

Example (H.W)

f (x) = sin (x), 0< x < π

PARSEVAL'S IDENTITY

If an and bn are the Fourier coefficients corresponding to f(x) and if f(x) satisfies the Dirichlet conditions

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$