$$a_0 = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \, d(\omega t) = \frac{V}{\pi}\left[-\cos \omega t\right]_0^{\pi} = \frac{2V}{\pi}$$

Next we determine a_n:

$$a_n = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \cos n\omega t \, d(\omega t)$$
$$= \frac{V}{\pi}\left[\frac{-n \sin \omega t \sin n\omega t - \cos n\omega t \cos \omega t}{-n^2+1}\right]_0^{\pi} = \frac{V}{\pi(1-n^2)}(\cos n\pi + 1)$$

With n even, $a_n = 2V/\pi(1-n^2)$; and with n odd, $a_n = 0$. However, this expression is indeterminate for $n = 1$, and therefore we must integrate separately for a_1.

$$a_1 = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \cos \omega t \, d(\omega t) = \frac{V}{\pi}\int_0^{\pi} \tfrac{1}{2}\sin 2\omega t \, d(\omega t) = 0$$

Now we evaluate b_n:

$$b_n = \frac{1}{\pi}\int_0^{\pi} V \sin \omega t \sin n\omega t \, d(\omega t) = \frac{V}{\pi}\left[\frac{n \sin \omega t \cos n\omega t - \sin n\omega t \cos \omega t}{-n^2+1}\right]_0^{\pi} = 0$$

Here again the expression is indeterminate for $n = 1$, and b_1 is evaluated separately.

$$b_1 = \frac{1}{\pi}\int_0^{\pi} V \sin^2 \omega t \, d(\omega t) = \frac{V}{\pi}\left[\frac{\omega t}{2} - \frac{\sin 2\omega t}{4}\right]_0^{\pi} = \frac{V}{2}$$

Then the required Fourier series is

$$f(t) = \frac{V}{\pi}\left(1 + \frac{\pi}{2}\sin \omega t - \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t - \frac{2}{35}\cos 6\omega t - \cdots\right)$$

The spectrum, **Fig. 3**, shows the strong fundamental term in the series and the rapidly decreasing amplitudes of the higher harmonics.

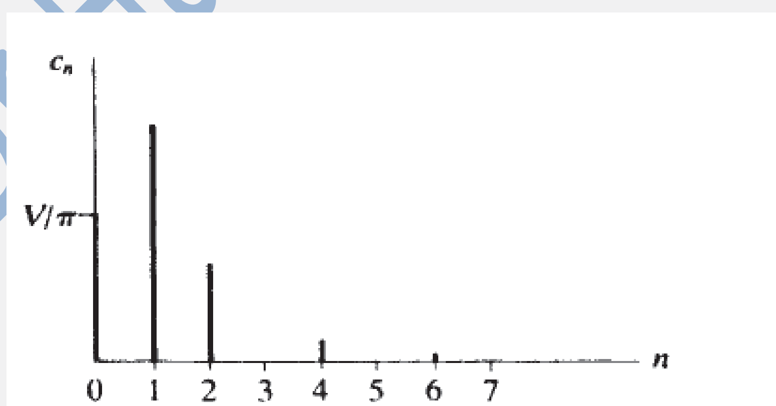


Fig.3

Sheet No 1

1. Prove $\int_{-L}^{L} \sin\frac{k\pi x}{L}\,dx = \int_{-L}^{L} \cos\frac{k\pi x}{L}\,dx = 0 \qquad \text{if } k = 1, 2, 3, \ldots.$

2. (a) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \text{Period} = 10$$

(b) Write the corresponding Fourier series.

(c) How should f(x) be defined at x = -5; x = 0; and x= 5 in order that the Fourier series will Converge to f(x) for **-5 ≤ x ≤ 5?**

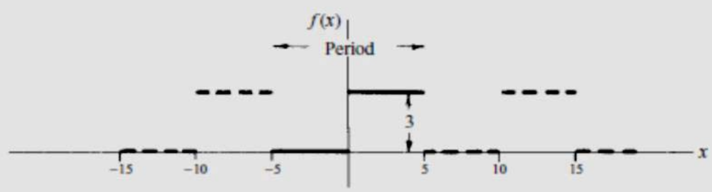


Fig. 13-6

(*a*) Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c + 2L$ as -5 to 5, so that $c = -5$. T

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx$$
$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx$$
$$= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \qquad \text{if } n \neq 0$$

$$\text{If } n = 0,\ a_n = a_0 = \frac{3}{5}\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3.$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$
$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx$$
$$= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi}$$

(*b*) The corresponding Fourier series is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)=\frac{3}{2}+\sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}$$
$$=\frac{3}{2}+\frac{6}{\pi}\left(\sin\frac{\pi x}{5}+\frac{1}{3}\sin\frac{3\pi x}{5}+\frac{1}{5}\sin\frac{5\pi x}{5}+\cdots\right)$$

(*c*) Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all continuity and to $\frac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x=-5$, 0, and 5, which of discontinuity, the series converges to $(3+0)/2=3/2$ as seen from the graph. If we redefi follows,

$$f(x)=\begin{cases}3/2 & x=-5\\ 0 & -5<x<0\\ 3/2 & x=0\\ 3 & 0<x<5\\ 3/2 & x=5\end{cases}\qquad \text{Period}=10$$

then the series will converge to $f(x)$ for $-5\leqq x\leqq 5$.

3.

Expand $f(x)=x^2, 0<x<2\pi$ in a Fourier series if (*a*) the period is 2π, (*b*) the period is not specified.

(a) The graph of f(x) with period 2π is shown in **Fig. 2**

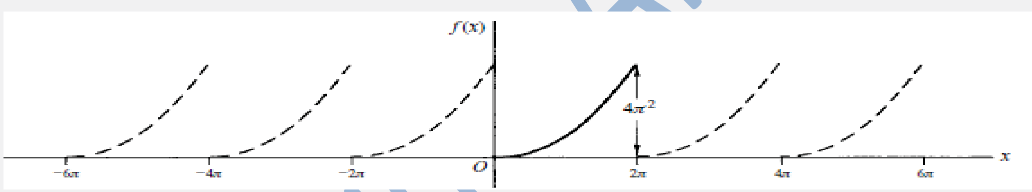


Fig. 2

Period $=2L=2\pi$ and $L=\pi$. Choosing $c=0$, we have

$$a_n=\frac{1}{L}\int_c^{c+2L}f(x)\cos\frac{n\pi x}{L}\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\,dx$$
$$=\frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi}=\frac{4}{n^2},\qquad n\neq 0$$

If $n=0$, $a_0=\frac{1}{\pi}\int_0^{2\pi}x^2\,dx=\frac{8\pi^2}{3}$.

$$b_n=\frac{1}{L}\int_c^{c+2L}f(x)\sin\frac{n\pi x}{L}\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nx\,dx$$
$$=\frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right)-(2x)\left(-\frac{\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi}=\frac{-4\pi}{n}$$

Then $f(x)=x^2=\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right)$.

This is valid for $0<x<2\pi$. At $x=0$ and $x=2\pi$ the series converges to $2\pi^2$.

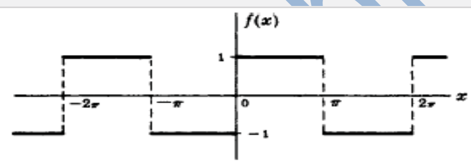
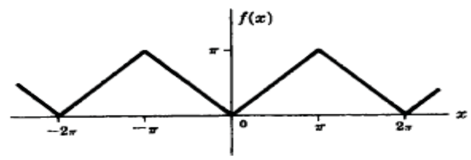
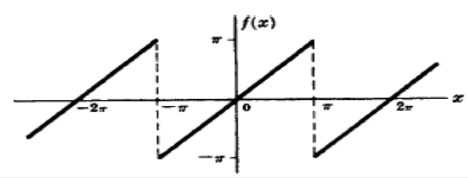
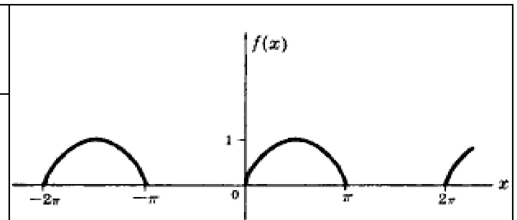
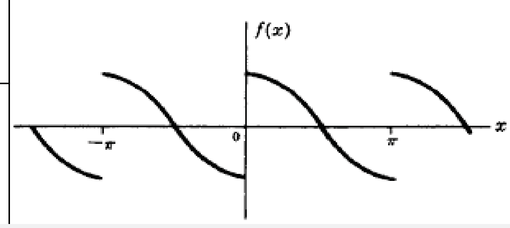
(*b*) If the period is not specified, the Fourier series cannot be determined uniquely in general.

4. Expand $f(x) = \sin x, 0 < x < \pi$, in a Fourier cosine series.

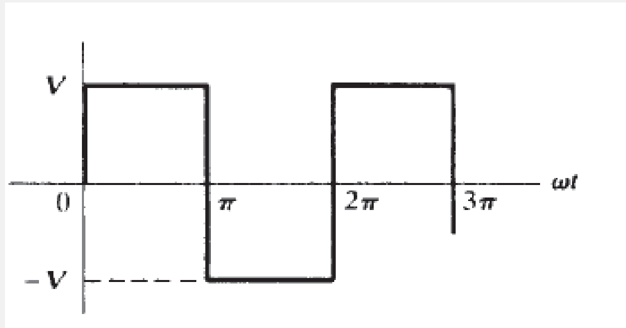
Ans:

$$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1+\cos n\pi)}{n^2-1}\cos nx$$
$$= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right)$$

5. **For the following graph find the Fourier series**

Ans: $\frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$	 Fig.
Ans: $\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$	 Fig.
Ans: $2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$	
Ans: $\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left(\frac{\cos 2x}{1\bullet 3} + \frac{\cos 4x}{3\bullet 5} + \frac{\cos 6x}{5\bullet 7} + \cdots\right)$	
Ans: $\frac{8}{\pi}\left(\frac{\sin 2x}{1\bullet 3} + \frac{2\sin 4x}{3\bullet 5} + \frac{3\sin 6x}{5\bullet 7} + \cdots\right)$	

6. Find the trigonometric Fourier series for the square wave shown in Fig. below and plot the line spectrum.



Ans:

$$f(t) = \frac{4V}{\pi}\sin\omega t + \frac{4V}{3\pi}\sin 3\omega t + \frac{4V}{5\pi}\sin 5\omega t + \cdots$$

7. Find the exponential Fourier series for the triangular wave shown in Figs. 1 and 2 and sketch the line spectrum.

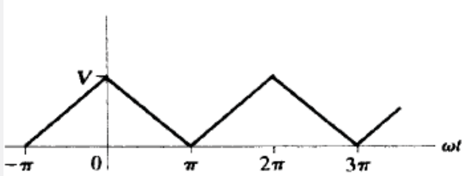


Fig.1

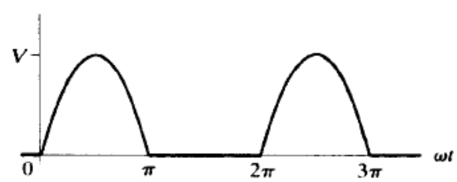


Fig.2

Sol: For Fig.1

In the interval $-\pi < \omega t < 0$, $f(t) = V + (V/\pi)\omega t$; and for $0 < \omega t < \pi$, $f(t) = V - (V/\pi)\omega t$. The wave is even and therefore the $\mathbf{A}_n$ coefficients will be pure real. By inspection the average value is $V/2$.

$$\mathbf{A}_n = \frac{1}{2\pi}\left\{\int_{-\pi}^{0}[V + (V/\pi)\omega t]e^{-jn\omega t}\,d(\omega t) + \int_{0}^{\pi}[V - (V/\pi)\omega t]e^{-jn\omega t}\,d(\omega t)\right\}$$
$$= \frac{V}{2\pi^2}\left[\int_{-\pi}^{0}\omega t e^{-jn\omega t}\,d(\omega t) + \int_{0}^{\pi}(-\omega t)e^{-jn\omega t}\,d(\omega t) + \int_{-\pi}^{\pi}\pi e^{-jn\omega t}\,d(\omega t)\right]$$
$$= \frac{V}{2\pi^2}\left\{\left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t - 1)\right]_{-\pi}^{0} - \left[\frac{e^{-jn\omega t}}{(-jn)^2}(-jn\omega t - 1)\right]_{0}^{\pi}\right\} = \frac{V}{\pi^2 n^2}(1 - e^{jn\pi})$$

For even n, $e^{jn\pi} = +1$ and $\mathbf{A}_n = 0$; for odd n, $\mathbf{A}_n = 2V/\pi^2 n^2$. Thus the series is

$$f(t) = \cdots + \frac{2V}{(-3\pi)^2}e^{-j3\omega t} + \frac{2V}{(-\pi)^2}e^{-j\omega t} + \frac{V}{2} + \frac{2V}{(\pi)^2}e^{j\omega t} + \frac{2V}{(3\pi)^2}e^{j3\omega t} + \cdots$$

The harmonic amplitudes

$$c_0 = \frac{V}{2} \qquad c_n = 2|\mathbf{A}_n| = \begin{cases} 0 & (n = 2, 4, 6, \ldots) \\ 4V/\pi^2 n^2 & (n = 1, 3, 5, \ldots) \end{cases}$$

For **Fig.2**

In the interval $0 < \omega t < \pi$, $f(t) = V \sin \omega t$; and from π to 2π, $f(t) = 0$. Then

$$\mathbf{A}_n = \frac{1}{2\pi}\int_0^{\pi} V \sin \omega t \, e^{-jn\omega t}\, d(\omega t)$$

$$= \frac{V}{2\pi}\left[\frac{e^{-jn\omega t}}{(1-n^2)}(-jn \sin \omega t - \cos \omega t)\right]_0^{\pi} = \frac{V(e^{-jn\pi}+1)}{2\pi(1-n^2)}$$

For even n, $\mathbf{A}_n = V/\pi(1-n^2)$; for odd n, $\mathbf{A}_n = 0$. However, for $n = 1$, the expression for $\mathbf{A}_n$ becomes indeterminate. L'Hôpital's rule may be applied; in other words, the numerator and denominator are separately differentiated with respect to n, after which n is allowed to approach 1, with the result that $\mathbf{A}_1 = -j(V/4)$.

The average value is

$$A_0 = \frac{1}{2\pi}\int_0^{\pi} V \sin \omega t \, d(\omega t) = \frac{V}{2\pi}\Big[-\cos \omega t\Big]_0^{\pi} = \frac{V}{\pi}$$

Then the exponential Fourier series is

$$f(t) = \cdots - \frac{V}{15\pi}e^{-j4\omega t} - \frac{V}{3\pi}e^{-j2\omega t} + j\frac{V}{4}e^{-j\omega t} + \frac{V}{\pi} - j\frac{V}{4}e^{j\omega t} - \frac{V}{3\pi}e^{j2\omega t} - \frac{V}{15\pi}e^{j4\omega t} - \cdots$$

The harmonic amplitudes,

$$c_0 = A_0 = \frac{V}{\pi} \qquad c_n = 2|\mathbf{A}_n| = \begin{cases} 2V/\pi(n^2-1) & (n = 2, 4, 6, \ldots) \\ V/2 & (n = 1) \\ 0 & (n = 3, 5, 7, \ldots) \end{cases}$$