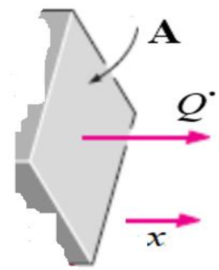
One-Dimensional Heat Conduction Equation

Consider a ***thin element*** of thickness x in a large ***plane wall***. Assume the density of the wall is **ρ**, the specific heat is ***C***, and the area of the wall normal to the direction of heat transfer is ***A***. An ***energy balance*** on this thin element during a small time interval **Δ*t*** can be expressed as:

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

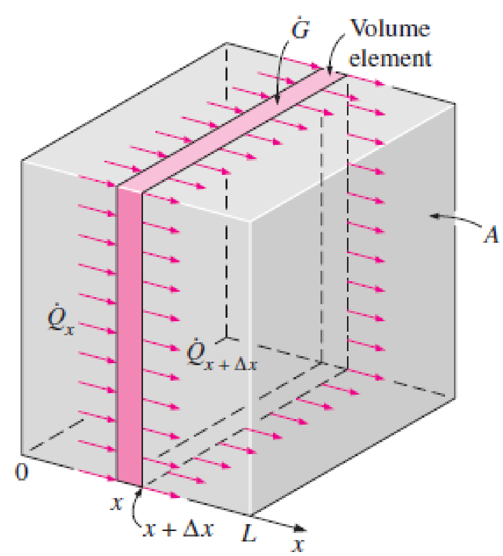
or:



$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

But the ***change in the energy*** content of the element and the ***rate of heat generation*** within the element can be expressed as:

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C A \Delta x (T_{t+\Delta t} - T_t)$$
$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} A \Delta x$$



$A_x = A_{x+\Delta x} = A$

Volume of element (m^3): V = $A\Delta x$
Mass of element (kg): $m = \boldsymbol{\rho}.V = \rho A \Delta x$

Substitute to get,

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{g} A \Delta x = \rho C A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Dividing by AΔx gives,

$$-\frac{1}{A}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Taking the limit as $\Delta x \to 0$ and $\Delta t \to 0$ yields

$$\lim_{\Delta x \to 0} \left(\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} \right) = \frac{\partial}{\partial x}(\dot{Q_x}) \quad , \qquad \lim_{\Delta t \to 0} \left(\frac{T_{t+\Delta t} - T_t}{\Delta t} \right) = \frac{\partial T}{\partial t}$$

And from ***Fourier law*** of heat conduction:

$$\dot{Q}_x = -kA\frac{\partial T}{\partial x},$$ The equation becomes:

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t}$$

or $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t}$$ [***Variable thermal conductivity***]

The ***thermal conductivity*** in most practical applications can be assumed to remain ***constant*** at some average value, so that:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ [***Constant thermal conductivity***] ***(x, y, z) (r, ϕ, z) (r, ϕ, θ)***

where the property $\alpha = k/\rho.C$ is the ***thermal diffusivity*** **(m^2/s)** of the material and represents ***how fast heat propagates through a material.***
This equation represents one ***dimensional heat conduction equation***. It reduces to the following forms under specified conditions:

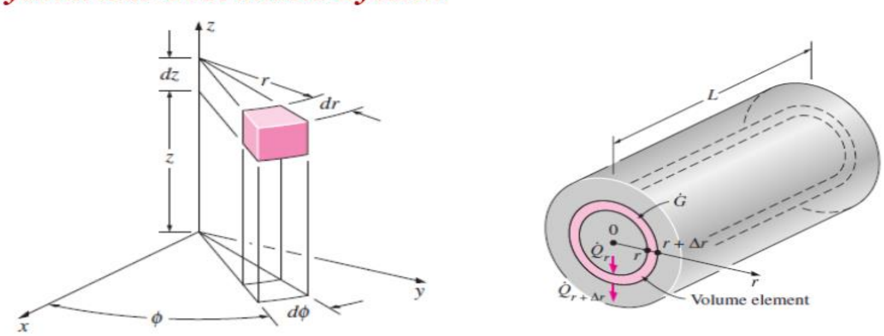
In the same way the ***one-dimensional heat conduction equation*** in cylindrical and spherical coordinate systems can be found. The ***rectangular***, ***cylindrical***, and ***spherical*** coordinate systems for the case of ***constant thermal conductivities*** are expressed as:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ **Rectangular coordinate system *(x, y, z)***

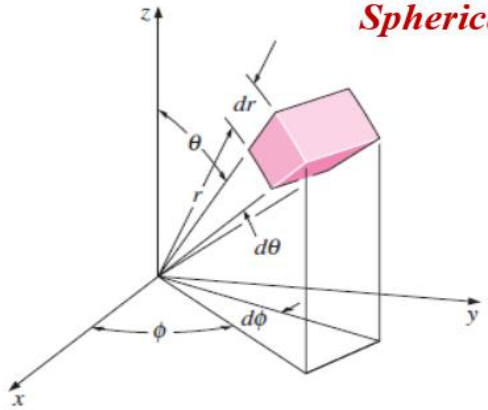
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ **Cylindrical coordnate system *(r, ϕ, z)***

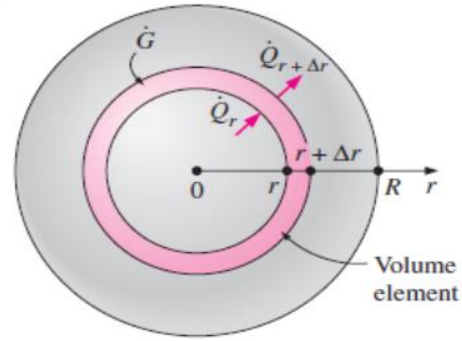
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ **Spherical coordinate system *(r, ϕ, θ)***

Cylindrical Coordinate System



Spherical coordinate system





For steady state with heat generation case the above equation be,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = 0$$

and for steady state without heat generation case the above equation be:

$$\frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$

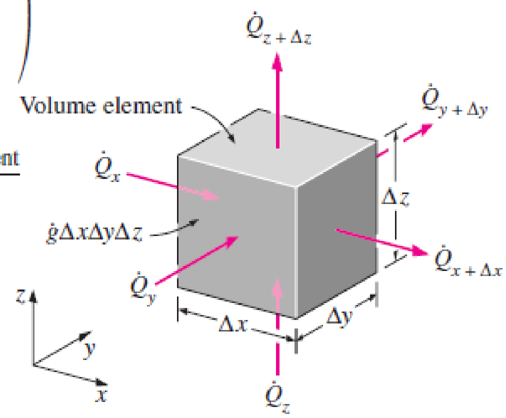
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 0$$

Solving these equations with the ***boundary conditions*** give the ***temperature distribution*** and ***heat transfer*** for any problems.

General Heat Conduction Equation (3dimensional) with Rectangular Coordinates

Consider a small rectangular element of ***length Δx***, ***width Δy***, and ***height Δz***. Assume the ***density*** of the body is ***ρ*** and the ***specific heat*** is ***C***. An ***energy balance*** on this element during a small ***time interval Δt*** can be expressed as:

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction at}\\ x,\ y,\ \text{and } z\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x,\\ y+\Delta y, \text{ and } z+\Delta z\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$



$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

Noting that the volume of the element is $V_{\text{element}}=\Delta x\Delta y\Delta z$, the ***change in the energy content*** of the element and the ***rate of heat generation*** within the element can be expressed as:

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C\Delta x\Delta y\Delta z(T_{t+\Delta t} - T_t)$$
$$\dot{G}_{\text{element}} = \dot{g}V_{\text{element}} = \dot{g}\Delta x\Delta y\Delta z$$

Substituting, we get:

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{g}\Delta x\Delta y\Delta z = \rho C\Delta x\Delta y\Delta z\frac{T_{t+\Delta t} - T_t}{\Delta t}$$

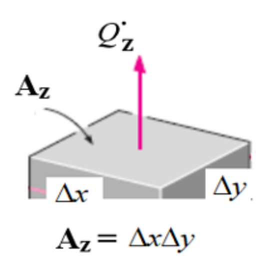
Dividing by **ΔxΔyΔz** gives:

$$-\frac{1}{\Delta y\Delta z}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} - \frac{1}{\Delta x\Delta z}\frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} - \frac{1}{\Delta x\Delta y}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} + \dot{g} = \rho C\frac{T_{t+\Delta t} - T_t}{\Delta t}$$

Noting that the ***heat transfer areas of the element for heat conduction*** in the ***x, y,*** and ***z*** directions are, $A_x = \Delta y\Delta z$, $A_y = \Delta x\Delta z$, $A_z = \Delta x\Delta y$ respectively, and taking the limit as Δx, Δy, Δz and $\Delta t \to 0$ yields:

The equation becomes:

$$-\frac{1}{A_x}\frac{\partial}{\partial x}(\dot{Q}_x) = -\frac{1}{A_x}\frac{\partial}{\partial x}\left(-k A_x\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$



$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial}{\partial z}\right)+\dot{g}=\rho C\frac{\partial T}{\partial t}$$

Divide both sides by k , and, $\alpha = k/\rho.C$ is the ***thermal diffusivity*** of the material, and in the case of ***constant thermal conductivity*** the equation reduces to:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

This is the ***general transient three dimensions heat conduction with heat generation***.

In the case of ***transient three dimensions heat conduction without heat generation:***

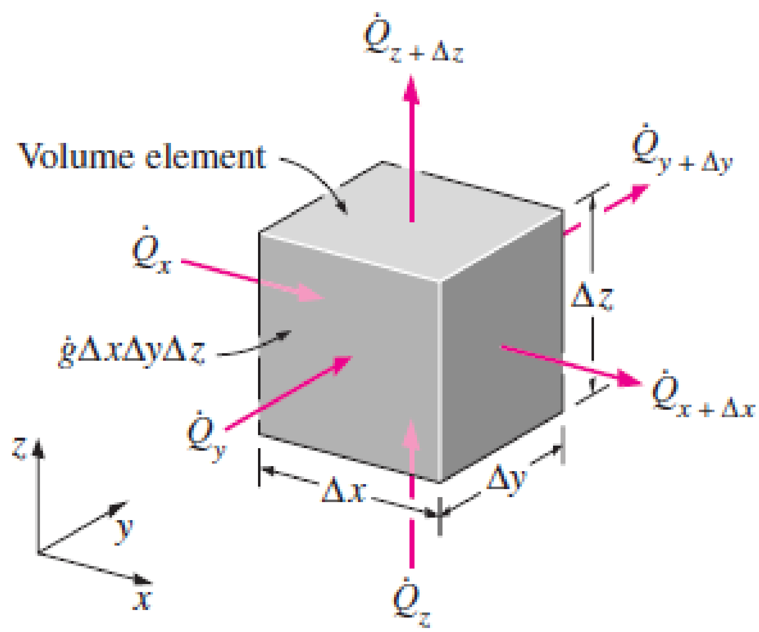
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$ (Called the ***diffusion equation***)

For ***Steady, three dimensions heat conduction with heat generation***.

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{g}}{k}=0$$ (Called the ***Poisson equation***)

And for ***Steady state, three dimensions heat conduction without heat generation:***

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$$ (Called the ***Laplace equation***)



Example-1:

Consider a **flat plate** solar collector placed at the roof of a house. The temperatures at the inner and outer surfaces of glass cover are measured to be **28°C** and **25°C**, respectively. The glass cover has a surface area of **2.2. m^2** and a thickness of **0.6 cm** and a thermal conductivity of **0.7 W/m·°C**. Heat is lost from the outer surface of the cover **by convection** and **radiation** with a convection heat transfer coefficient of **10 W/m^2·°C** and an ambient temperature of **15°C**. ***Determine the fraction of heat lost from the glass cover by radiation***.

Solution:

The glass cover of a flat plate solar collector with specified inner and outer surface temperatures is considered. The fraction of heat lost from the glass cover by radiation is to be determined.

Assumptions **1** Steady operating conditions exist since the surface temperatures of the glass remain constant at the specified values. **2** Thermal properties of the glass are constant.

Under steady conditions, the rate of heat transfer through the glass by conduction is

$$\dot{Q}_{\text{cond}} = kA\frac{\Delta T}{L} = (0.7\ \text{W/m}\cdot°\text{C})(2.2\ \text{m}^2)\frac{(28-25)°\text{C}}{0.006\ \text{m}} = 770\ \text{W}$$

The rate of heat transfer from the glass by convection is

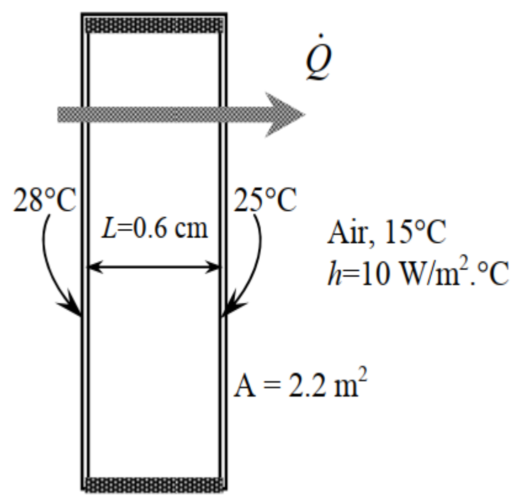
$$\dot{Q}_{\text{conv}} = hA\Delta T = (10\ \text{W/m}^2\cdot°\text{C})(2.2\ \text{m}^2)(25-15)°\text{C} = 220\ \text{W}$$

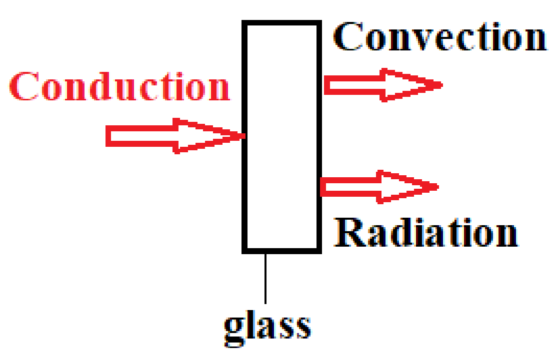
Under steady conditions, the heat transferred through the cover by conduction should be transferred from the outer surface by convection and radiation. That is,

$$\dot{Q}_{\text{rad}} = \dot{Q}_{\text{cond}} - \dot{Q}_{\text{conv}} = 770 - 220 = 550\ \text{W}$$

Then the fraction of heat transferred by radiation becomes

$$f = \frac{\dot{Q}_{\text{rad}}}{\dot{Q}_{\text{cond}}} = \frac{550}{770} = \mathbf{0.714} \quad (\text{or } 71.4\%)$$





Example-2:

A 1.4 m long, 0.2cm diameter **electrical wire** extends across a room that is maintained at **20 °C**. **Heat is generated in** the wire as a result of resistance heating, and the surface temperature of the wire is measured to be **240 °C** in steady operation. Also, the **voltage** drop and electric **current** through the wire are measured to be **110 V** and **3 A**, respectively. **Disregarding** any heat transfer by **radiation**, ***determine the convection heat transfer coefficient for heat transfer between the outer surface of the wire and the air in the room.***

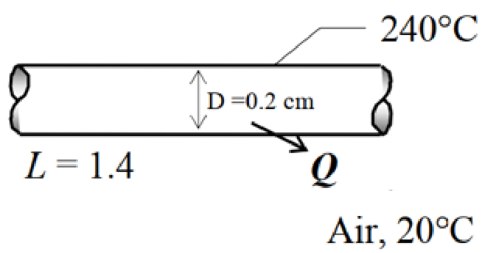
Solution:

The convection heat transfer coefficient for heat transfer from an electrically heated wire to air is to be determined by measuring temperatures when steady operating conditions are reached and the electric power consumed.

Assumptions 1 Steady operating conditions exist since the temperature readings do not change with time. **2** Radiation heat transfer is negligible.

In steady operation, the rate of heat loss from the wire equals the rate of heat generation in the wire as a result of resistance heating. That is,

$$\dot{Q} = \dot{E}_{\text{generated}} = VI = (110\ \text{V})(3\ \text{A}) = 330\ \text{W}$$



The surface area of the wire is

$$A_s = (\pi D)L = \pi(0.002\ \text{m})(1.4\ \text{m}) = 0.00880\ \text{m}^2$$

The Newton's law of cooling for convection heat transfer is expressed as

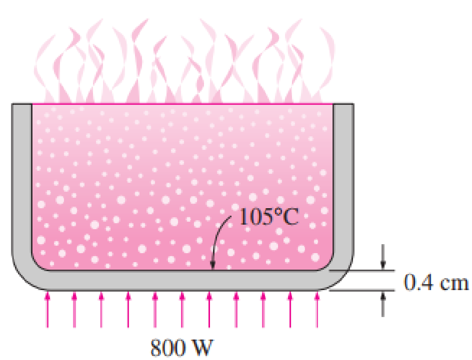
$$\dot{Q} = hA_s(T_s - T_\infty)$$

Disregarding any heat transfer by radiation , the convection heat transfer coefficient is determined to be

$$h = \frac{\dot{Q}}{A_s(T_1 - T_\infty)} = \frac{330\ \text{W}}{(0.00880\ \text{m}^2)(240-20)°\text{C}} = \mathbf{170.5\ W/m^2.°C}$$

H.W.

Q: An aluminum pan whose thermal conductivity is 237 W/m·°C has a flat bottom with diameter 20 cm and thickness 0.4 cm. Heat is transferred steadily to boiling water in the pan through its bottom at a rate of 800 W. If the inner surface of the bottom of the pan is at 105°C, ***determine the temperature of the outer surface of the bottom of the pan.***



Answer: **T = 105.43 °C**