Solving System of Linear Equations

6.1 Linear Equation

$y = mx$ is an equation, in which variable y is expressed in terms of x and the constant m, is called Linear Equation. In Linear Equation exponents of the variable is always 'one'.

6.2 Linear Equation in n variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + ... + a_nx_n = b$$

Where $x_1, x_2, x_3, ..., x_n$ are variables and

$a_1, a_2, a_{13}, ..., a_n$ and b are constants.

6.3 System of Linear Equations:

A Linear System of m linear equations and n unknowns is:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ... + a_{3n}x_n = b_3$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m$$

Where $x_1, x_2, x_3, ..., x_n$ are variables or unknowns and a's and b's are constants.

6.4 Augmented Matrix

System of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_{31} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_{31} = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_{31} = b_3$$

Can be written in the form of matrices product

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Or we may write it in the form AX=b,

Where A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, b = $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Augmented matrix is $[A:b] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

Example 6.1:

Write the matrix and augmented form of the system of linear equations

3x – y + 6z = 6

x + y + z = 2

2x + y+4z = 3

Solution: Matrix form of the system is

$$\begin{bmatrix} 3 & -1 & 6 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

Augmented form is $[A:b] = \begin{bmatrix} 3 & -1 & 6 & 6 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

6.5 Methods for Solving System of Linear Equations

1. **Gaussian Elimination Method**
2. **Gauss -Jorden Elimination Method**

6.5.1 Gaussian Elimination.

Gaussian elimination is a general method of finding possible solutions to a linear system of equations.

Gaussian Elimination Method

Step 1. By using elementary row operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & A_{12} & A_{13} & B_1 \\ 0 & 1 & A_{23} & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

Step 2. Find solution by back – substitutions.

Example 6.2:

Solve the system of linear equations by Gaussion- Elimination method

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ 2x_1 - x_2 - 2x_3 = 6 \\ 4x_1 + 2x_2 + 3x_3 = 7 \end{cases}$$

Solution:

Step 1.

Augmented matrix is	
$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3 \\ 2 & -1 & -2 & 6 \\ 4 & 2 & 3 & 7 \end{array}\right]$	$R_2 = r_2 - 2r_1$ $R_3 = r_3 - 4r_1$
$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3 \\ 0 & -3 & -4 & 0 \\ 0 & -2 & -1 & -5 \end{array}\right]$	$R_3 = 3r_3 - 2r_2$
$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3 \\ 0 & -3 & -4 & 0 \\ 0 & 0 & 5 & -15 \end{array}\right]$	$R_2 = -r_2$ $R_3 = \frac{r_3}{5}$
$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right]$	

Equivalent system of equations form is:

Step 2. Back Substitution

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 3x_2 + 4x_3 &= 0 \\ x_3 &= -3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_3 &= -3 \\ x_2 &= -4x_3/3 = 12/3 = 4 \\ x_1 &= 3 - x_2 - x_3 = 3 - 4 + 3 = 2 \end{aligned}$$

Solutions are $x_1 = 2,\ x_2 = 4,\ x_3 = -3$

Example 6.3:

For the below figure calculate the values of the unknown flow rates F_1, F_2 and F_3 by using Gaussion- Elimination method

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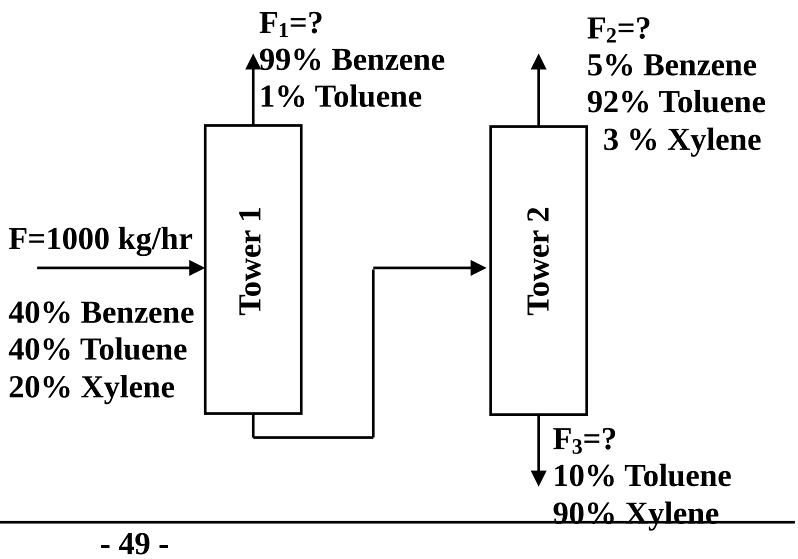

Component material balance gives these three equations of three variables

$$F_1 + \quad F_2 + \quad F_3 = 1000$$
$$0.99F_1 + 0.05F_2 + 0F_3 = 400$$
$$0.01F_1 + 0.92F_2 + 0.1F_3 = 400$$

Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1000 \\ 0.99 & 0.05 & 0 & 400 \\ 0.01 & 0.92 & 0.1 & 400 \end{bmatrix} \qquad \begin{matrix} R_2=r_2-(0.99/1)\times r_1 \\ R_3=r_3-(0.01/1)\times r_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1000 \\ 0 & -0.94 & -0.99 & -590 \\ 0 & 0.91 & 0.09 & 390 \end{bmatrix} \qquad R_3=r_3-(0.91/(-0.94))r_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1000 \\ 0 & -0.94 & -0.99 & -590 \\ 0 & 0 & -0.8684 & -181.17 \end{bmatrix} \qquad \begin{matrix} R_2=r_2/(-0.94) \\ R_3=r_3/(-0.8684) \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1000 \\ 0 & 1 & 1.0532 & 627.6596 \\ 0 & 0 & 1 & 208.6253 \end{bmatrix}$$

Equivalent system of equations form is:

$F_1 + F_2 + F_3 = 1000$

$F_2 + 1.0532F_3 = 627.6596$

$F_3 = 208.6253$

Step 2. Back Substitution

$F_3 = 208.6253$

$F_2 = 627.6596-1.0532F_3 = 627.6596-1.0532\times 208.6253 = 407.9354$

$F_1 = 1000- F_2-F_3 = 1000 -208.6253 - 407.9354= 383.4393$

6.5.2 Gauss - Jorden Elimination Method

Gauss - Jorden Method
By using elementary row operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & B_1 \\ 0 & 1 & 0 & B_2 \\ 0 & 0 & 1 & B_3 \end{bmatrix}$$

Example 6.4:
Solve the system of linear equations by Gauss-Jorden elimination method
$x_1 + x_2 + 2x_3 = 8$
$- x_1 - 2x_2 + 3x_3 = 1$
$3x_1 -7x_2 + 4x_3 = 10$
Solution:
Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \quad R_2= r_2+r_1 \quad R_3= r_3-3r_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \quad R_2=-r_2 \quad R_3=r_3-10r_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \quad R_3=-r_3/52$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1=r_1-2r_3 \quad R_2=r_2+5r_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1=r_1-r_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Equivalent system of equations form is:
$x_1 = 3$

$x_2 = 1$
$x_3 = 2$
is the solution of the system.

Example 6.5:
Total and component material balance on a system of distillation columns gives the flowing equations:-

$$F_1 + F_2 + F_3 + F_4 = 1690$$
$$0.4F_1 + 0.15F_2 + 0.25F_3 + 0.2F_4 = 412.5$$
$$0.25F_1 + 0.8F_2 + 0.3F_3 + 0.45F_4 = 701$$
$$0.08F_1 + 0.05F_2 + 0.45F_3 + 0.3F_4 = 487.3$$

Use Gauss - Jorden method to compute the four un-known's in above equations:-
Solution:
Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1690 \\ 0.4 & 0.15 & 0.25 & 0.2 & 412.5 \\ 0.25 & 0.8 & 0.3 & 0.45 & 701 \\ 0.08 & 0.05 & 0.45 & 0.3 & 487.3 \end{bmatrix}$$

$R_2 = r_2-(0.4/1)r_1$
$R_3 = r_3-(0.25/1)r_1$
$R_4 = r_4-(0.08/1)r_1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1690 \\ 0 & -0.25 & -0.15 & -0.2 & -263.5 \\ 0 & 0.55 & 0.05 & 0.2 & 278.5 \\ 0 & -0.03 & 0.37 & 0.22 & 352.1 \end{bmatrix}$$

$R_3 = r_3-(0.55/(-0.25))r_2$
$R_4 = r_4-((-0.03)/(-0.025))r_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1690 \\ 0 & -0.25 & -0.15 & -0.2 & -263.5 \\ 0 & 0 & -0.28 & -0.24 & -301.2 \\ 0 & 0 & 0.388 & 0.244 & 383.72 \end{bmatrix}$$

$R_4 = r_4-((0.0388)/(-0.028))r_3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1690 \\ 0 & -0.25 & -0.15 & -0.2 & -263.5 \\ 0 & 0 & -0.28 & -0.24 & -301.2 \\ 0 & 0 & 0 & -0.08857 & -33.657 \end{bmatrix}$$

$R_2 = r_2/(-0.25)$
$R_3 = r_3/(-0.028)$
$R_4 = r_4/(-0.0887)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1690 \\ 0 & 1 & 0.6 & 0.8 & 1054 \\ 0 & 0 & 1 & 0.85714 & 1075.74 \\ 0 & 0 & 0 & 1 & 380 \end{bmatrix}$$

$R_1 = r_1 - r_4$
$R_2 = r_2 - (0.8/1)r_4$
$R_3 = r_3 - (0.85714/1)r_4$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1310 \\ 0 & 1 & 0.6 & 0 & 750 \\ 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 0 & 1 & 380 \end{bmatrix}$$

$R_1 = r_1 - r_3$
$R_2 = r_2 - ((-0.6)/1)r_3$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 560 \\ 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 0 & 1 & 380 \end{bmatrix}$$

$R_1 = r_1 - r_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 260 \\ 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 0 & 1 & 380 \end{bmatrix}$$

Equivalent system of equations form:
$F_1 = 260$, $F_2 = 300$, $F_3 = 750$ and
$F_4 = 380$ is the solution of the system.

Example 6.6

Balance the following chemical equation:

$x_1\ P_2I_4 + x_2\ P_4 + x_3\ H_2O \rightarrow x_4\ PH_4I + x_5\ H_3PO_4$

<u>Solution:</u>

P balance: $2x_1 + 4x_2 = x_4 + x_5$
I balance: $4x_1 = x_4 + x_5$
H balance: $2x_3 = 4x_4 + 3x_5$
O balance: $x_3 = 4x_5$

Re-write these as homogeneous equations, each having zero on its right hand side:

$2x_1 + 4x_2 - x_4 - x_5 = 0$
$4x_1 - x_4 - x = 0$
$2x_3 - 4x_4 - 3x_5 = 0$
$x_3 - 4x_5 = 0$

At this point, there are four equations in five unknowns. To complete the system, we define an auxiliary equation by arbitrarily choosing a value for one of the coefficients:

$x_1 = 1$

We can easily solve the above equations to balance this reaction using MATLAB such in table 6.1

Table (6.1) Matlab code and results for solution example (6.6)	

Matlab Code	`A = [2 4 0 -1 -1`
	`4 0 0 -1 0`
	`0 0 2 -4 -3`
	`0 0 1 0 -4`
	`1 0 0 0 0];`
	`B= [0;0;0;0;1];`
	`X = A\B`

Results	`X =`
	`1.0000`
	`1.3000`
	`12.8000`
	`4.0000`
	`3.2000`

This does not yield integral coefficients, but multiplying by 10 will do the trick: The balanced equation will be:

$10\ P_2I_4 + 13\ P_4 + 128\ H_2O \rightarrow 40\ PH_4I + 32\ H_3PO_4$