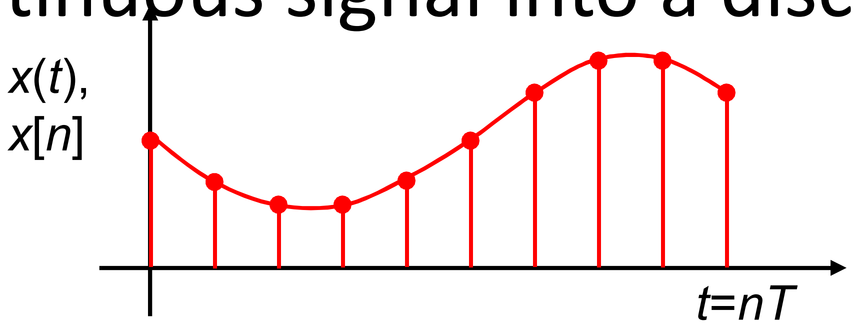
Digital Signal Processing

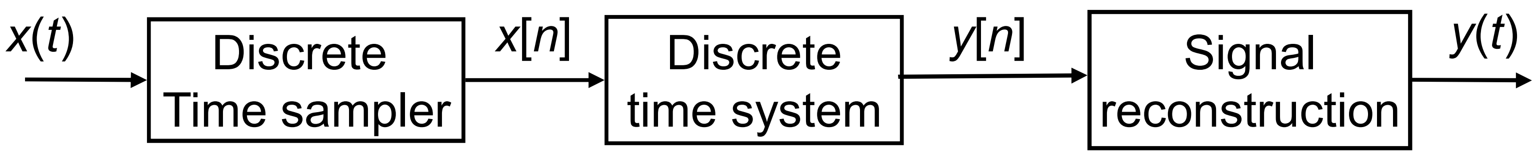
Sampling Theorem

What is Discrete Time Sampling?

Sampling is the transformation of a continuous signal into a discrete signal

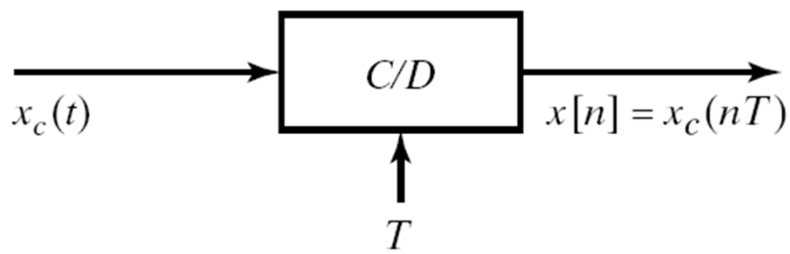


T is the sampling period



Sampling

- In this method x[n] obtained from $x_c(t)$ according to the relation :



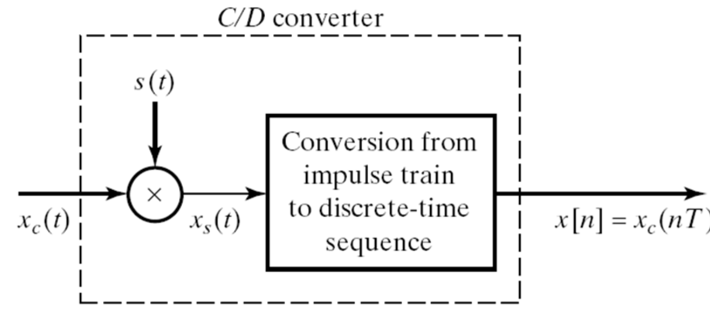
$$x[n] = x_c(nT) \quad -\infty < n < \infty$$

$$T \rightarrow sampling\ \ period \quad f_s = 1/T \rightarrow sampling\ frequency$$

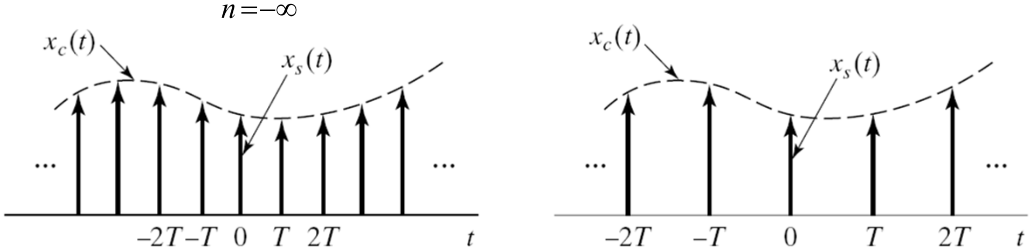
Sampling with a Periodic Impulse Train

- Figure(a) is not a representation of any physical circuits, but it is convenient for gaining insight in both the time and frequency domain.

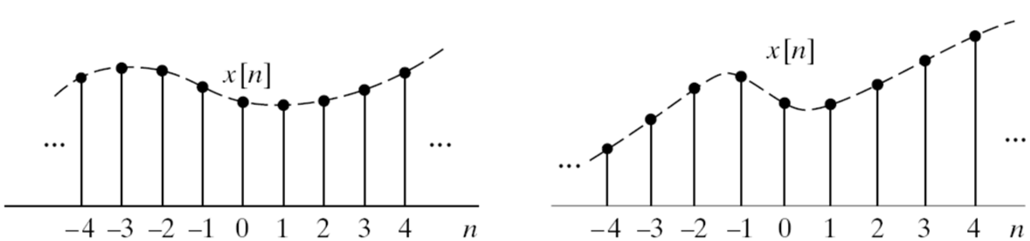
$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



(a)



(b)



(c)

(a) Overall system

(b) $x_s(t)$ for two sampling rates

(c) Output for two sampling rates

Frequency Domain Representation of Sampling

$$x_s(t) = x_c(t)s(t) = x_c(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT) \quad (Modulation)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t-nT) \quad (Shifting\ \ property)$$

- Let us now consider the Fourier transform of $x_s(t)$:
- If $s(t) \xleftrightarrow{Fourier} S(j\Omega)$ and $x_c(t) \xleftrightarrow{Fourier} X_C(j\Omega)$

$S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)$ where $\Omega_s = 2\pi / T$ is the sampling rate in radians/s.

$$X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j(\Omega - k\Omega_s)\right)$$

Frequency Domain Representation of Sampling

- By applying the continuous-time Fourier transform to equation

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT)$$

We obtain

$$X_S(j\Omega) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega Tn}$$

$$x[n] = x_c(nT) \quad and \quad X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

consequently

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T} = X(e^{j\Omega T}) \Rightarrow X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

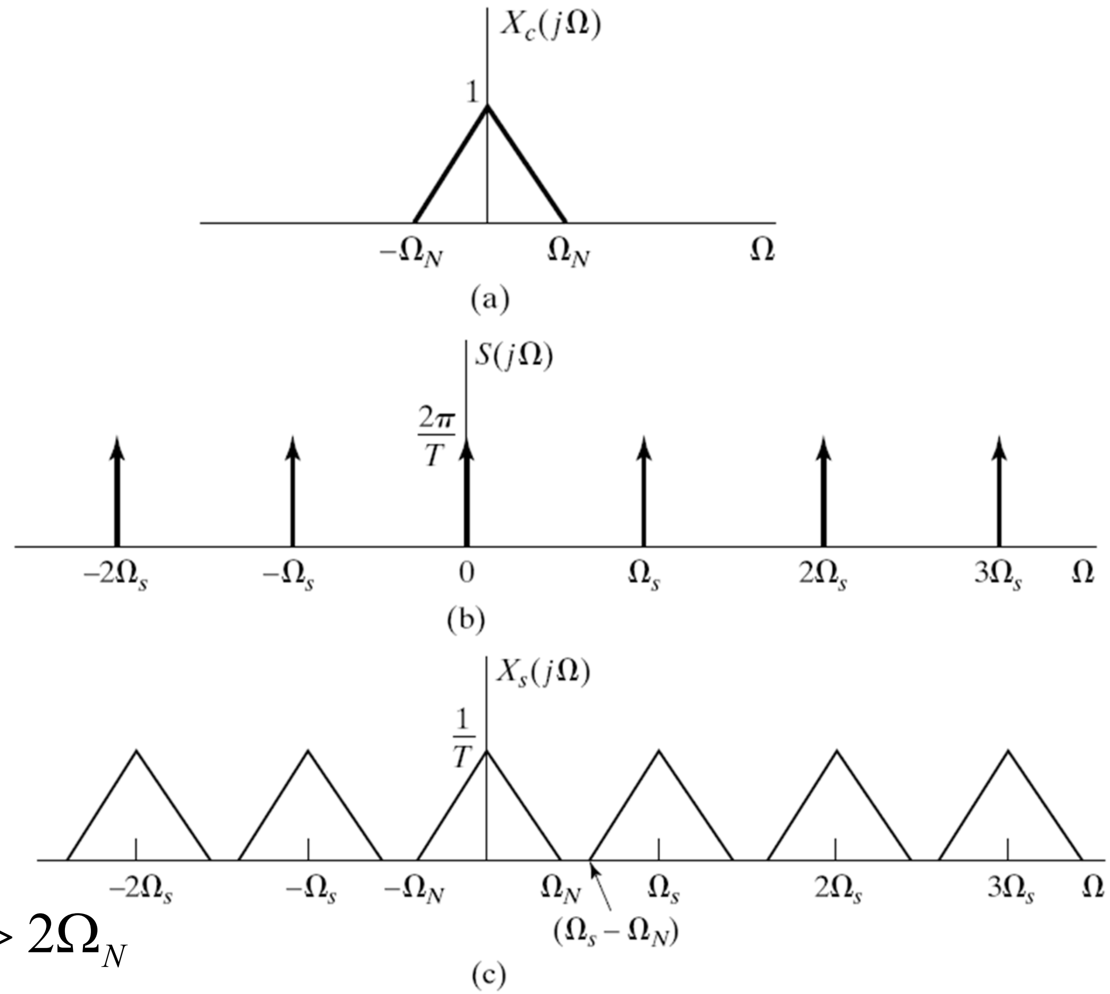
Exact Recovery of Continuous-Time from Its Samples

- (a) represents a band limited Fourier transform of $x_c(t)$ Whose highest nonzero frequency is Ω_N.

- (b) represents a periodi impulse train with Ω_S frequency.

- (c) shows the output of impulse modulator in the case

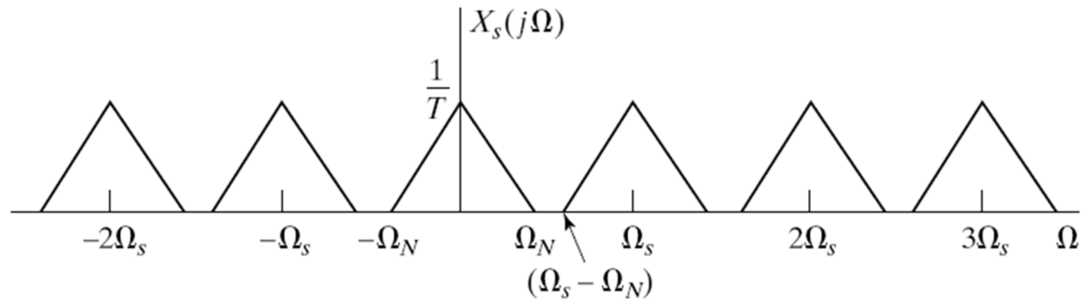
$$\Omega_S - \Omega_N > \Omega_N \Rightarrow \Omega_S > 2\Omega_N$$



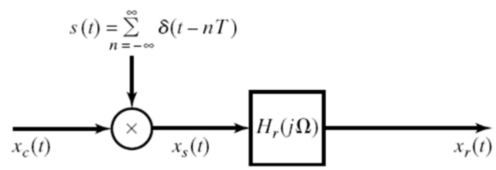
Exact Recovery of Continuous-Time from Its Samples

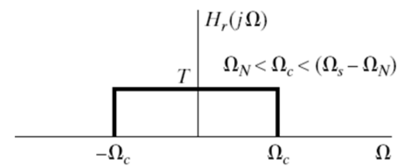
- In this case $X_C(j\Omega)$ don't overlap
- therefore $x_c(t)$ can be recovered from $x_s(t)$ with an ideal low pass filter $H_r(j\Omega)$ with gain T and cutoff frequency
- It means $\Omega_N < \Omega_C < \Omega_S - \Omega_N$

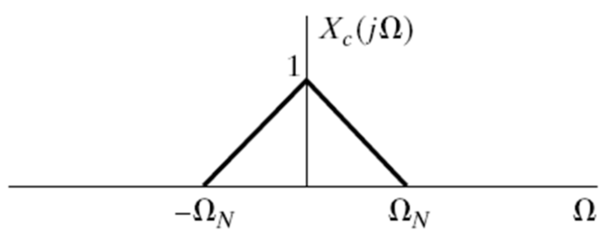
$$X_r(j\Omega) = X_C(j\Omega)$$



(c)







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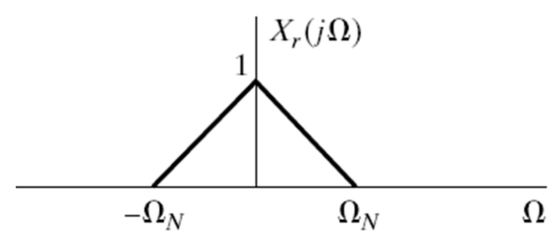
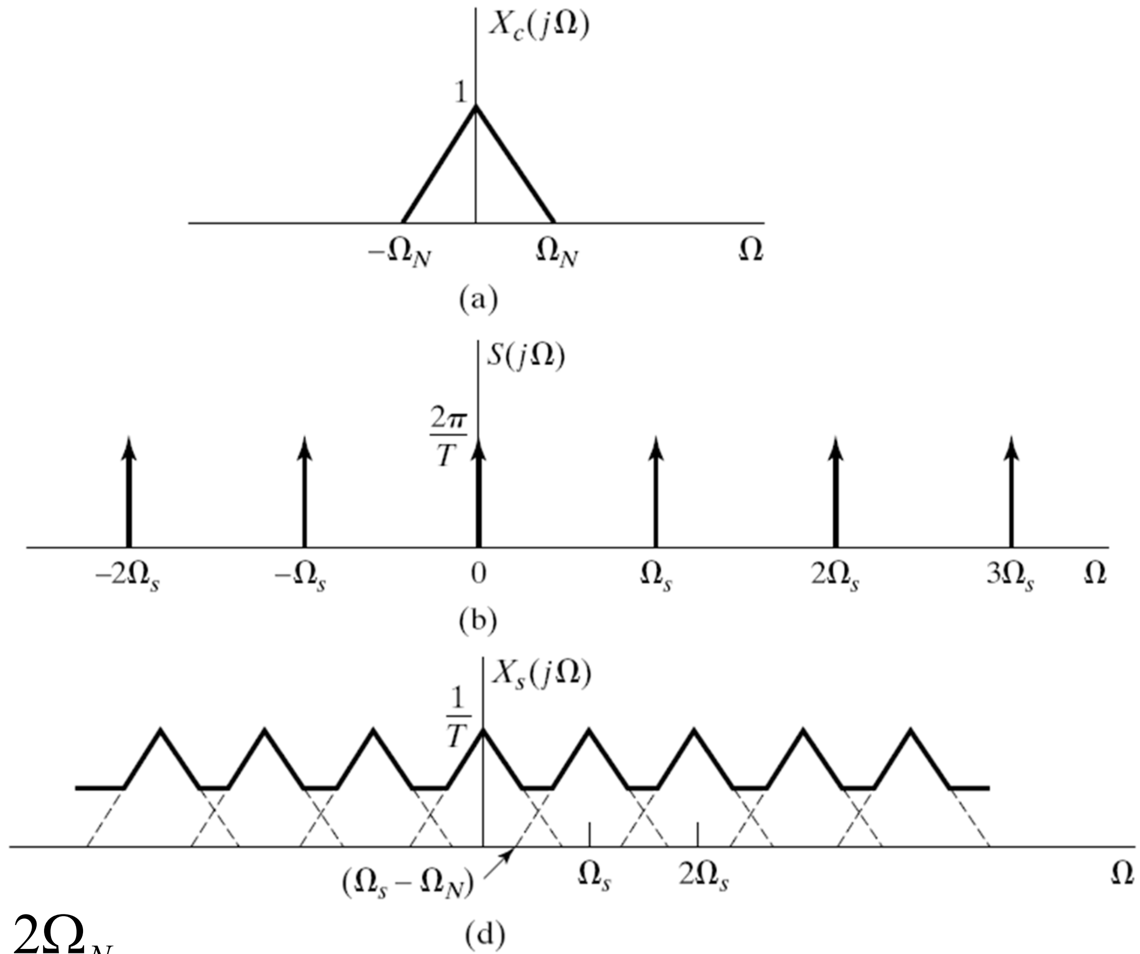


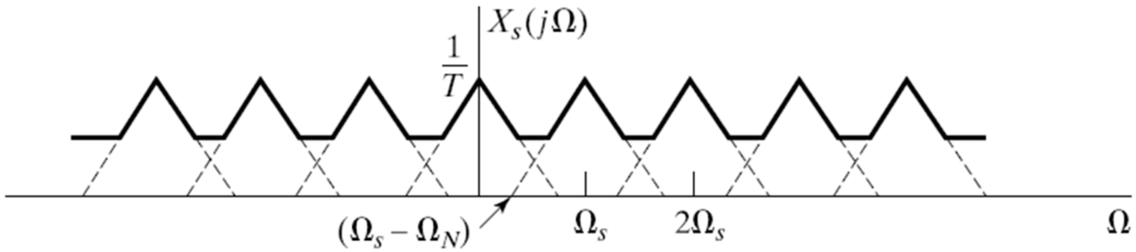
Aliasing Distortion

- (a) represents a band limited Fourier transform of $x_c(t)$ Whose highest nonzero frequency is Ω_N.
- (b) represents a periodic impulse train with frequency. Ω_S
- (c) shows the output of impulse modulator in the case

$$\Omega_S - \Omega_N < \Omega_N \Rightarrow \Omega_S < 2\Omega_N$$



Aliasing Distortion



(d)

- In this case the copies of $X_C(j\Omega)$ overlap and is not longer recoverable by lowpass filtering therefore the reconstructed signal is related to original continuous-time signal through a distortion referred to as aliasing distortion.