

Interpolation and Approximation

What is interpolation?

Many times, data is given only at discrete points such as $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), (x_{n+1}, y_{n+1})$. So, how then does one find the value of y at any other value of x ?

Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ point (Figure 2.1). Then we can find the value of y at any other value of x . This is called **interpolation**.

Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation, but instead, is called **extrapolation**.

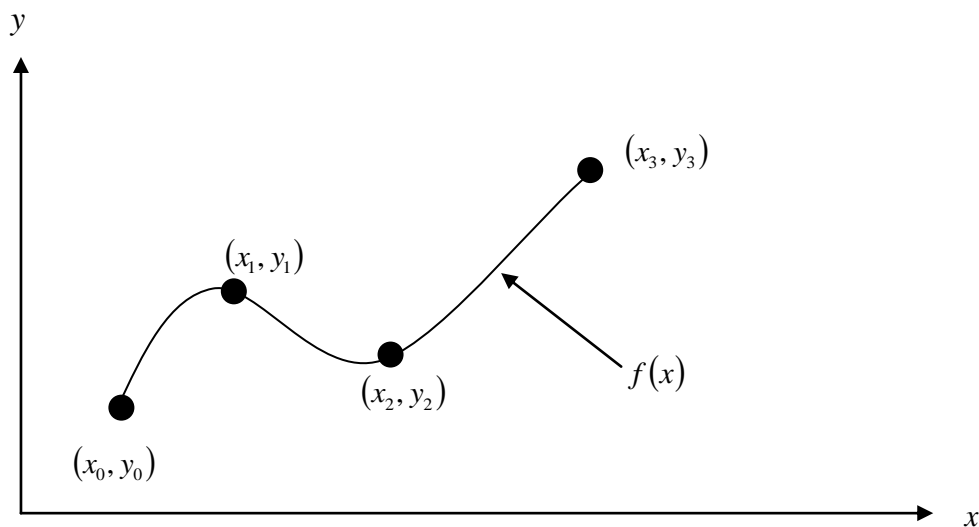


Figure 2.1 Interpolation of discrete data

For $n+1$ data points, there is one and only one polynomial of order n that passes through all the points. For example, there is only one straight line (that is, a first-order polynomial) that connects two points. Similarly, only one parabola connects a set of three points.

Polynomial Interpolation consists of determining the unique n^{th} order polynomial that fits $n+1$ data points. This polynomial then provides a formula to compute intermediate values.

One of the methods used to find this polynomial is called the Lagrange method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method.

2.1 Lagrange Interpolating Polynomial

Consider a function $f(x)$ that passes through the two distinct points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ as shown in Figure 2.2. The first order polynomial that approximates the function between these two points can be expressed as

$$f(x) = a + bx$$

Where a and b are constants. $f(x)$ can also be written in Lagrangian form as

$$f(x) = c_0(x - x_1) + c_1(x - x_0)$$

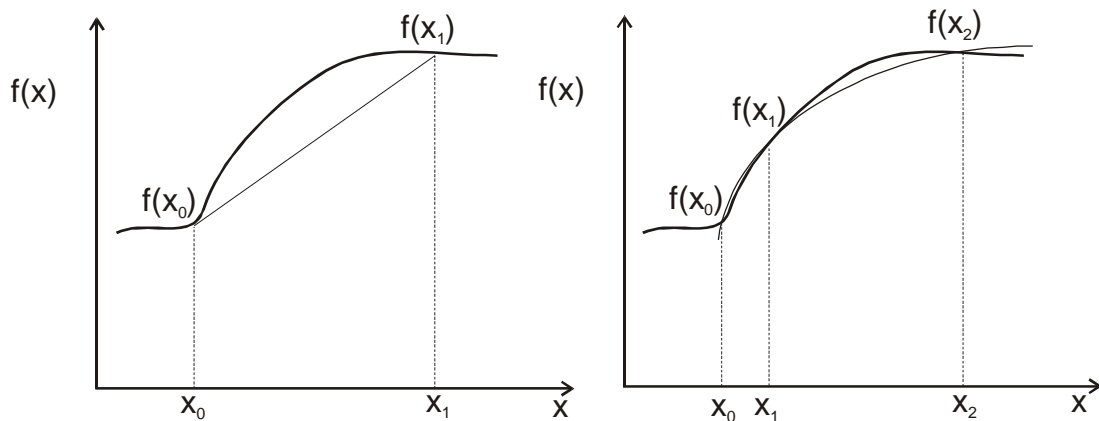


Figure 2.2 First and second order polynomial approximation.

I. Linear

By weighting the average of the two values used to produce the coordinates of the line the formula:

$$f_1(x) = L_1 f(x_1) + L_2 f(x_2) \quad \text{where:} \quad L_1 = \frac{x - x_2}{x_1 - x_2} \quad L_2 = \frac{x - x_1}{x_2 - x_1}$$

Example 2.1

Compute a 4-decimal place value of $\ln 9.2$ from $\ln 9.0 = 2.1972$, $\ln 9.5 = 2.2513$ by linear Lagrange interpolation and determine the error, using the exact value of $\ln 9.2 = 2.2192$.

Solution:

$x_1 = 9.0$, $x_2 = 9.5$, $f_1 = \ln 9.0 = 2.1972$, $f_2 = \ln 9.5 = 2.2513$; hence we get

$$L_1(x) = \frac{x - 9.5}{-0.5} = -2.0(x - 9.5), \quad L_1(9.2) = -2.0(-0.3) = 0.6$$

$$L_2(x) = \frac{x - 9.0}{0.5} = 2.0(x - 9.0), \quad L_2(9.2) = 2 \times 0.2 = 0.4$$

$$\ln 9.2 \approx p_1(9.2) = L_1(9.2)f_1 + L_2(9.2)f_2 = 0.6 \times 2.1972 + 0.4 \times 2.2513 = 2.2188$$

The absolute error is $|2.2192 - 2.2188| = 0.0004$

II. Quadratic

By weighting the average of the three points that produce the parabola we can derive the formula:

$$f_2(x) = L_1 f(x_1) + L_2 f(x_2) + L_3 f(x_3)$$

where:

$$L_1 = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \quad L_2 = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \quad L_3 = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

Example 2.2

Compute $\ln 9.2$ from the data in the previous example 2.1 and the additional third value $\ln 11.0 = 2.3979$.

Solution:

$$L_1(x) = \frac{(x-9.5)(x-11.0)}{(9.0-9.5)(9.0-11.0)} = x^2 - 20.5x + 104.5 \quad \Rightarrow L_1(9.2) = 0.5400$$

$$L_2(x) = \frac{(x-9.0)(x-11.0)}{(9.5-9.0)(9.5-11.0)} = -\frac{1}{0.75}(x^2 - 20x + 99) \quad \Rightarrow L_2(9.2) = 0.4800$$

$$L_3(x) = \frac{(x-9.0)(x-9.5)}{(11.0-9.0)(11.0-9.5)} = \frac{1}{3}(x^2 - 18.5x + 85.5) \quad \Rightarrow L_3(9.2) = -0.0200$$

$$\ln 9.2 \approx p_2(9.2) = 0.5400 \times 2.1972 + 0.4800 \times 2.2513 - 0.0200 \times 2.3979 = 2.2192.$$

The absolute error is $|2.2192 - 2.2192| = 0.0000$

III. General Lagrange Interpolating Polynomial

In general, the Lagrange polynomial can be represented as:

$$f_{n-1}(x) = \sum_{i=1}^n L_i(x) f(x_i) \quad \text{where} \quad L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^n \frac{x-x_j}{x_i-x_j}$$

Example 2.3

Find the Lagrange interpolation polynomial that takes the values prescribed below

x_k	0	1	2	4
$f(x_k)$	1	1	2	5

Solution

$$P_3(x) = \sum_{k=0}^3 L_{3,k}(x) f(x_k)$$

$$P_3(x) = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(1) \\ + \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(2) + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(5)$$

When working with grids having large numbers of intervals one typically assigns a set of low degree ($n = 1, 2,$ or 3) basis functions to each adjacent set of $n+1 = 2, 3,$ or 4 nodes.

Example 2.4

The following table gives the value of density of saturated water for various temperatures of saturated steam.

Temp °C (= T)	100	150	200	250
Density kg/m ³ (= d)	958	917	865	799

- 1) Use third order Lagrange interpolating polynomials to correlate density as a function of temperature.
- 2) Find the densities when the temperatures are 130°C .

Solution

$$L_1 = \frac{(T-150)(T-200)(T-250)}{(100-150)(100-200)(100-250)} = -1.3333 \times 10^{-6} T^3 + 4 \times 10^{-3} T^2 - 0.1566 T + 10$$

$$L_2 = \frac{(T-100)(T-200)(T-250)}{(150-100)(150-200)(150-250)} = 4 \times 10^{-6} T^3 - 2.2 \times 10^{-3} T^2 + 0.38 T - 20$$

$$L_3 = \frac{(T-100)(T-150)(T-250)}{(200-100)(200-150)(200-250)} = -4 \times 10^{-6} T^3 + 2 \times 10^{-3} T^2 - 0.31 T + 15$$

$$L_4 = \frac{(T-100)(T-150)(T-200)}{(250-100)(250-150)(250-200)} = 1.3333 \times 10^{-6} T^3 + 6 \times 10^{-4} T^2 - 0.08666 T - 4$$

$$f_4(x) = L_1 f(x_1) + L_2 f(x_2) + L_3 f(x_3) + L_4 f(x_4) = -4 \times 10^{-6} T^3 - 4 \times 10^{-4} T^2 - 0.53 T + 1019$$

$$f_4(130) = 934.5520$$

Example 2.5

Use Lagrange global interpolation by one polynomial and piecewise polynomial interpolation with quadratic for the following nodes.

x_k	0	1	2	4	5
$f(x_k)$	0	16	48	88	0

Solution

Global interpolation by one polynomial: $P(x) = \sum_{k=0}^4 L_{4,k}(x) f(x_k)$

$$\begin{aligned} P_4(x) &= \frac{(x-1)(x-2)(x-4)(x-5)}{(0-1)(0-2)(0-4)(0-5)} (0) + \frac{(x-0)(x-2)(x-4)(x-5)}{(1-0)(1-2)(1-4)(1-5)} (16) \\ &+ \frac{(x-0)(x-1)(x-4)(x-5)}{(2-0)(2-1)(2-4)(2-5)} (48) + \frac{(x-0)(x-1)(x-2)(x-5)}{(4-0)(4-1)(4-3)(4-5)} (88) + 0 \\ &= -4.6667x^4 + 33.33x^3 - 59.3333x^2 + 46.6667x \end{aligned}$$

Piecewise polynomial interpolation with quadratic

$$P_2(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)}(0) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(16) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(48); \quad 0 \leq x \leq 2$$
$$= 8x + 8x^2$$

$$P_2(x) = \frac{(x-4)(x-5)}{(2-4)(2-5)}(48) + \frac{(x-2)(x-5)}{(4-2)(4-5)}(88) + \frac{(x-2)(x-4)}{(5-2)(5-4)}(0); \quad 2 \leq x \leq 5$$
$$= -280 + 236x - 36x^2$$

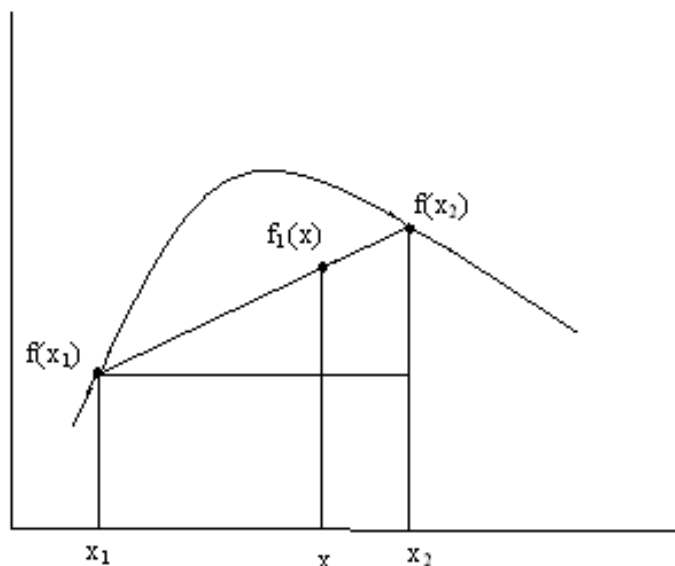
2.1 Newton Divided Difference Interpolating

The Lagrangian interpolation polynomials are useful in discussions on numerical integration. An alternative in interpolation is ‘*Newton’s Divided Difference Interpolation*’. It involves fewer arithmetical operations.

Another advantage of Newton’s rests with the following scenario. Suppose we need to improve the accuracy and increase the number of grid points. From the forms of Lagrange interpolation polynomials, all the terms have to be evaluated once again, and this is a huge amount of work if the number of points is large. Newton’s does not suffer from this drawback, and just one additional term needs to be computed.

I. Linear Interpolation

Consider the diagram below in which a curve is modeled (poorly) by $\overline{x_1x_2}$:



Using similar triangles the slopes are the same and hence:

$$\frac{f_1(x) - f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

And thus the coordinate on the curve at x_1 can be approximated by rearranging the above to become:

$$f_1(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

Example 2.6

Estimate the common logarithm of 10 using linear Newton's interpolation.

(a) Interpolate between $\log 8 = 0.9030900$ and $\log 12 = 1.0791812$.

(b) Interpolate between $\log 9 = 0.9542425$ and $\log 11 = 1.0413927$.

For each of the interpolations, compute the percent relative error based on the true value.

Solution

$$\text{a) } f_1(10) = 0.90309 + \frac{1.0791812 - 0.90309}{12 - 8}(10 - 8) = 0.991136$$

$$\varepsilon_r = \frac{1 - 0.991136}{1} \times 100\% = 0.886\%$$

$$\text{b) } f_1(10) = 0.9542425 + \frac{1.0413927 - 0.9542425}{11 - 9}(10 - 9) = 0.997818$$

$$\varepsilon_r = \frac{1 - 0.997818}{1} \times 100\% = 0.218\%$$

II. Quadratic Interpolation

To reduce the error, a quadratic interpolation that introduces some curvature into the interpolation is used. The form:

$$f_2(x) = b_1 + b_2(x - x_1) + b_3(x - x_1)(x - x_2)$$

Let $x = x_1$ to produce:

$$b_1 = f(x_1)$$

Let $x = x_2$ and use the previous identity to produce:

$$f(x_2) = f(x_1) + b_2(x_2 - x_1) + b_3(x - x_1)(x - x_2) \Rightarrow b_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

And again by substitution of b_1 and b_2 we derive that:

$$f(x_3) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_3 - x_1) + b_3(x_3 - x_1)(x_3 - x_2) \Rightarrow b_3 = \frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1}$$

$$f_2(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + \frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1}(x - x_1)(x - x_2)$$

Example 2.7

Fit a second-order Newton's Interpolating polynomial to estimate $\log 10$ using the data from Example 2.7 at $x = 8, 9,$ and 11 . Compute the true percent relative error.

Solution

First, order the points

$$x_1 = 9 \quad f(x_1) = 0.9542425$$

$$x_2 = 11 \quad f(x_2) = 1.0413927$$

$$x_3 = 8 \quad f(x_3) = 0.9030900$$

$$b_1 = 0.9542425$$

$$b_2 = \frac{1.0413927 - 0.9542425}{11 - 9} = 0.0435751$$

$$b_3 = \frac{\frac{0.9030900 - 1.0413927}{8 - 11} - 0.0435751}{8 - 9} = \frac{0.0461009 - 0.0435751}{8 - 9} = -0.0025258$$

Substituting these values yields the quadratic formula

$$f_2(x) = 0.9542425 + 0.0435751(x - 9) - 0.0025258(x - 9)(x - 11)$$

which can be evaluated at $x = 10$ for

$$f_2(10) = 0.9542425 + 0.0435751(10 - 9) - 0.0025258(10 - 9)(10 - 11) = 1.0003434$$

$$\varepsilon_t = \frac{1 - 1.0003434}{1} \times 100\% = 0.03434\%$$

III. General form of Newton Divided Difference Interpolating Polynomial

In general, if we find the finite differences defined as:

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

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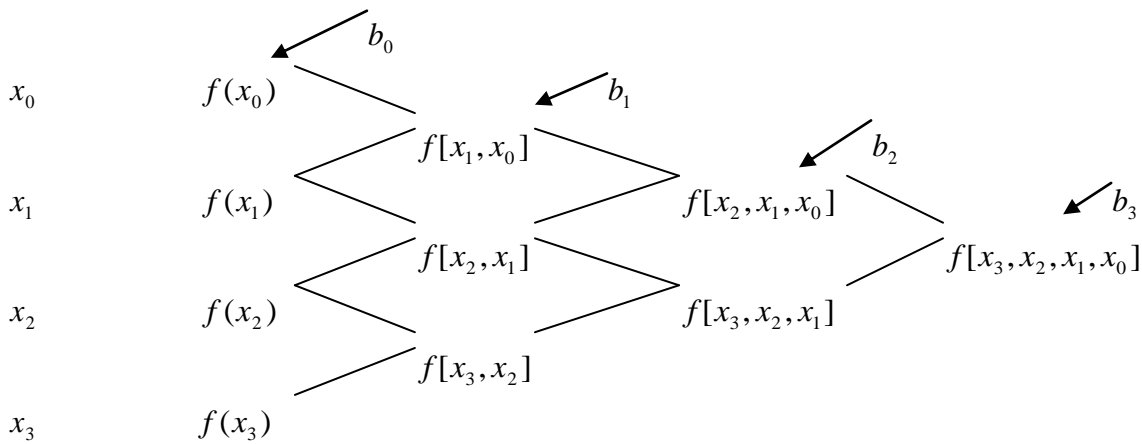
$$f[x_n, x_{n-1}, \dots, x_1] = \frac{f[x_n, x_{n-1}, \dots, x_2] - f[x_{n-1}, x_{n-2}, \dots, x_1]}{x_n - x_1}$$

Then the general Newton Interpolating Polynomial of order $n - 1$ with n data points is defined as:

$$f_{n-1}(x) = b_1 + b_2(x - x_1) + b_3(x - x_1)(x - x_2) + \dots + b_n(x - x_1)(x - x_2)\dots(x - x_{n-1})$$

$$\begin{aligned}
 b_1 &= f(x_1) \\
 b_2 &= f[x_2, x_1] \\
 b_3 &= f[x_3, x_2, x_1] \\
 &\vdots \\
 b_n &= f[x_n, x_{n-1}, \dots, x_1]
 \end{aligned}$$

Where



For an example of a third order polynomial, given (x_0, y_0) , (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) ,

$$\begin{aligned}
 f_3(x) &= f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\
 &\quad + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)
 \end{aligned}$$

Example 2.8

The following table gives the value of density of saturated water for various temperatures of saturated stream.

Temp° C (= T)	:	100	150	200	250	300
Density kg/m ³ (= d)	:	958	917	865	799	712

Using Newton divided difference interpolating find the densities when the temperatures are 130°C and 275°C respectively.

Solution

i	T	D	$f[x_{i+1}, x_i]$	$f[x_{i+2}, x_{i+1}, x_i]$	$f[x_{i+3}, x_{i+2}, x_{i+1}, x_i]$	$f[x_{i+4}, x_{i+3}, x_{i+2}, x_{i+1}, x_i]$
1	100	958				
			-0.8200			
2	150	917		-0.0022		
			-1.0400		-4×10^{-6}	
3	200	865		-0.0028		-2.6667×10^{-8}
			-1.3200		-9.333×10^{-6}	
4	250	799		-0.0042		
			-1.7400			
5	300	712				

$$P_4 = 958 - 0.8200 \times (T - 100) - 0.0022 \times (T - 100) \times (T - 150) - 4 \times 10^{-6} \times (T - 100) \times (T - 150) \times (T - 200) - 2.6667 \times 10^{-8} \times (T - 100) \times (T - 150) \times (T - 200) \times (T - 250)$$

$$P_4 = 999 - 0.0167T - 0.0051T^2 + 1.4667 \times 10^{-5}T^3 - 2.6667 \times 10^{-8}T^4$$

$$P_4(130) = 934.6864 \text{ kg/m}^3$$

$$P_4(275) = 758.7187 \text{ kg/m}^3$$

Or by direct substitution

$$P_4(130) = 934.6864 \text{ kg/m}^3$$

$$\begin{aligned} P(130) &= 958 - 0.82 \times (130 - 100) - 0.0022 \times (130 - 100) \times (130 - 150) - 4 \times 10^{-6} \times (130 - 100) \times (130 - 150) \times (130 - 200) \\ &\quad - 2.6667 \times 10^{-8} \times (130 - 100) \times (130 - 150) \times (130 - 200) \times (130 - 250) \\ &= 934.6864 \text{ kg/m}^3 \end{aligned}$$

$$\begin{aligned} P(275) &= 958 - 0.82 \times (275 - 100) - 0.0022 \times (275 - 100) \times (275 - 150) - 4 \times 10^{-6} \times (275 - 100) \times (275 - 150) \times (275 - 200) \\ &\quad - 2.6667 \times 10^{-8} \times (275 - 100) \times (275 - 150) \times (275 - 200) \times (275 - 250) \\ &= 758.7188 \text{ kg/m}^3 \end{aligned}$$