Solved Problems – Polar Coordinates

Ex) Graph the following Polar equation curves:

① $r^2 = a^2 \cos 2\theta = 9\cos 2\theta$ (lemniscate)

Sol.) At the beginning we try to guess the proper "θ-step" from the following:

① The behaviour of polar curve is depending on the trigonometric function ($\cos 2\theta$), Hence

$0 \leq 2\theta \leq 2\pi \longrightarrow 0 \leq \theta \leq \pi \longrightarrow \text{Step}_1 = \frac{\pi}{4}$

② The max. and/or min. values of r & $r = 0$.

* $r^2 = 0 \longrightarrow 9\cos 2\theta = 0 \longrightarrow 2\theta = \pi/2, 3\pi/2, 5\pi/2$ & $7\pi/2$

$\therefore \theta = \pi/4, 3\pi/4, \frac{5\pi}{4}$ & $\frac{7\pi}{4}$

* $r^2_{max.} = 9 \times (1) = 9 \longrightarrow r = \mp 3$ when

$\cos 2\theta = 1 \longrightarrow 2\theta = 0, 2\pi$ & 4π

$\therefore \theta = 0, \pi$ & 2π

* $r^2_{min} \neq 9 * (-1) \neq -9$ (not defined in R)

③ Symmetry tests

$f(-\theta) = 9\cos(-2\theta) = 9\cos 2\theta = f(\theta)$

$f(\pi-\theta) = 9\cos(2\pi - 2\theta) = 9\cos 2\theta = f(\theta)$

$(-r)^2 = r^2$

$\therefore$ the Polar curve is symmetrical about x-axis, y-axis and about the origin.

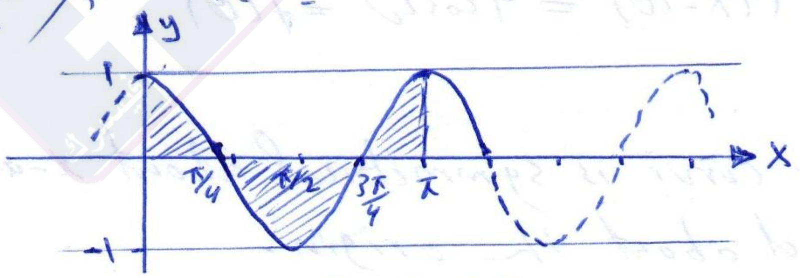
→

* أما أننا في الخطوات الثلاث السابقة حاولنا إيجاد أفضل تدريجة لقيم الزاوية θ الجدولية ، وذلك بالاعتماد على فهم تعريف متغير الدالة القطبية في المثال (١) وذلك من خلال فهم تعرف وطبيعة الدالة الأساسية المكونة للدالة القطبية وهي هنا الدالة المثلثية $(\cos 2\theta)$ وهي كما نعرف دالة دورية Periodic يمكننا ملاحظة جميع تغيراتها (max. & min. & القيم الصفرية) من خلال التدريجة $(\pi/4)$ ، (انظر الشكل أسفل) .

* في الخطوة الثانية قمنا (بالتمهيد) بفهم الدالة العكسية $(9\cos 2\theta)$ الطرفية (max. & min.) بالإضافة (إلى القيم الصفرية) .
أما في الخطوة الثالثة فإننا فحصنا حالات تناظر الدالة المؤثرة لتسهيل الرسم .

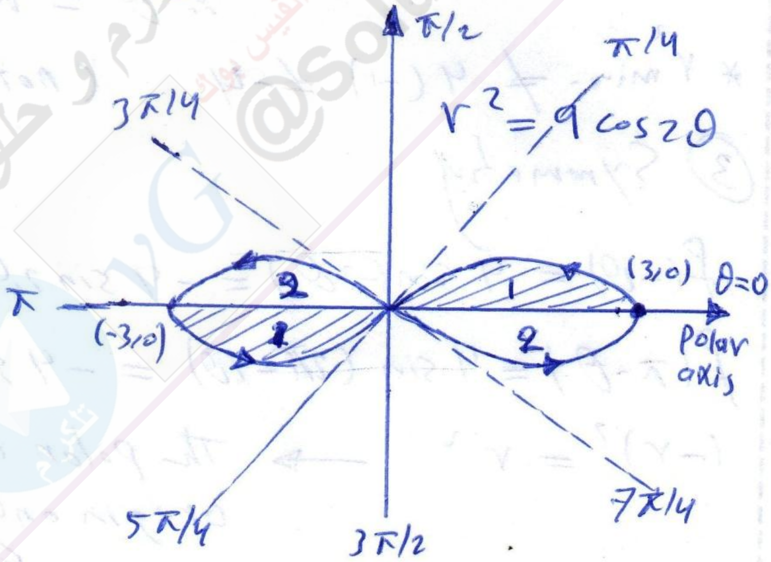
* إن الأدلة المشروحة أعلاه هي لتسهيل رسم منحنى أي دالة قطبية (Polar Equation) وذلك باختيار أكبر تدريجة زاوية $(\theta\text{-step})$ في الجدول $(r-\theta)$ دون فقدان أي من تفاصيل الرسم (Graph details) المهمة .

* وأخيراً فإننا نلاحظ أن قيم الزاوية (θ) المقابلة لقيم (r) العظمى (r_{max}) والقيم الصفرية $(r=0)$ هي من مضاعفات التدريجة المقترحة $(\theta\text{-step} = \pi/4)$ ولذلك تم اعتماد الأخيرة في الجدول :



$y = \cos 2x$

θ	$r^2 = 9\cos 2\theta$	$r = \mp 3\sqrt{\cos 2\theta}$
0	9	∓ 3
$\pi/4$	0	0
$\pi/2$	(-9) *	not defined in R
$3\pi/4$	0	0
π	9	∓ 3
$5\pi/4$	0	0
$3\pi/2$	(-9) *	not defined in R
$7\pi/4$	0	0
2π	9	∓ 3



② $r^2 = 4\sin 2\theta$ (lemniscate)

Sol.)

① Trigonometric function $(\sin 2\theta) \rightarrow$

$0 \leq 2\theta \leq 2\pi \rightarrow 0 \leq \theta \leq \pi \rightarrow (\theta\text{-step} = \frac{\pi}{4})$

② Extreme values of (r) & $(r=0)$:

$* \; r^2 = r = 0 \rightarrow 4\sin 2\theta = 0 \rightarrow$

$2\theta = 0, \pi, 2\pi, 3\pi \; \& \; 4\pi$

$\therefore \theta = 0, \pi/2, \pi, 3\pi/2 \; \& \; 2\pi$

$* \; r^2_{max.} = 4 * (1) = 4 \Rightarrow r = \mp 2$ when

$\sin 2\theta = 1 \rightarrow 2\theta = \pi/2 \; \& \; 5\pi/2$

$\therefore \theta = \pi/4 \; \& \; 5\pi/4$

$* \; r^2_{min.} \neq 4(-1) \neq -4$ (not defined in R)

③ Symmetry

$f(-\theta) = 4\sin(-2\theta) = -4\sin 2\theta \neq f(\theta)$

$f(\pi-\theta) = 4\sin(2\pi - 2\theta) = -4\sin 2\theta \neq f(-\theta)$

$(-r)^2 = r^2 \rightarrow$ The polar curve is sym. about origin only.

$\rightarrow$ The proper θ-step is $(\pi/4)$

θ	$r^2 = 4\sin 2\theta$	$r = \mp 2\sqrt{\sin 2\theta}$
0	0	0
$\pi/4$	4	∓ 2
$\pi/2$	0	0
$3\pi/4$	(-4) *	Not defined in R
π	0	0
$5\pi/4$	4	∓ 2
$3\pi/2$	0	0
$7\pi/4$	(-4) *	Not defined in R
2π	0	0

** ~~~~~~~~~~~~~~~~ **

③ $r = a + b\cos\theta = 1 + 2\cos\theta$; limacon with inner loop.

Sol.)

① Trigonometric function $(\cos\theta) \rightarrow$

$0 \le \theta \le 2\pi \rightarrow \left(\theta\text{-step} = \frac{2\pi}{4} = \frac{\pi}{2}\right)$

② $(r_{max.}, r_{min.}$ & $r = 0) \rightarrow$

$- r = 0 \rightarrow 1 + 2\cos\theta = 0 \rightarrow \cos\theta = -1/2 \rightarrow$

$\theta = 2\pi/3 \ \& \ 4\pi/3$

$- r_{max.} = 1 + 2(1) = 3$ when $\cos\theta = 1 \rightarrow$

$\theta = 0 \ \& \ 2\pi$

$- r_{min.} = 1 + 2(-1) = -1$ when $\cos\theta = -1 \rightarrow$

$\theta = \pi$

③ Symmetry

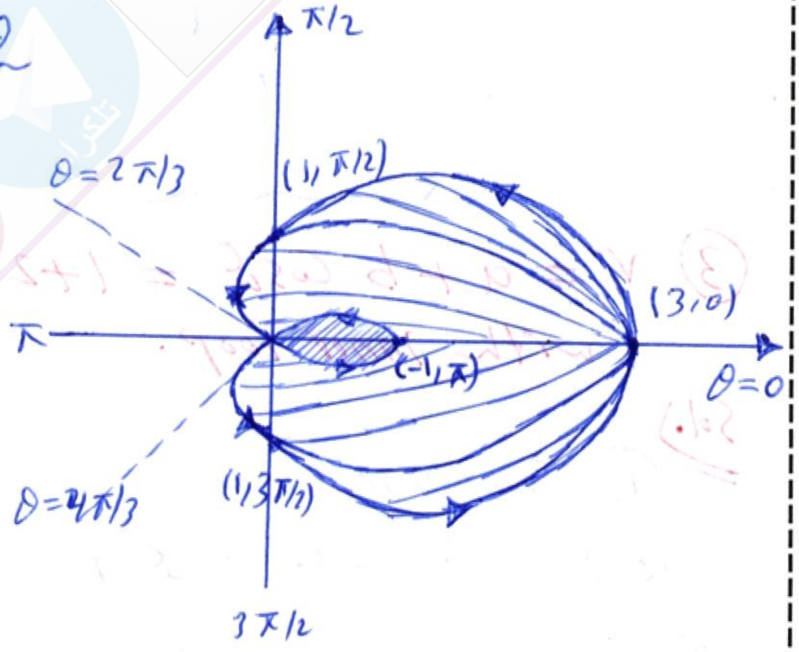
$f(-\theta) = 1 + 2\cos(-\theta) = 1 + 2\cos\theta = f(\theta)$

$f(\pi - \theta) = 1 + 2\cos(\pi - \theta) = 1 - 2\cos\theta \neq f(-\theta)$

$-r \neq r \longrightarrow$ Polar Curve is sym. about X-axis only.

∴ The Proper θ-step is $(\pi/2)$ but should including $(2\pi/3 \ \& \ 4\pi/3)$.

θ	$r = 1 + 2\cos\theta$
0	3
$\pi/2$	1
$2\pi/3$	0
π	-1
$4\pi/3$	0
$3\pi/2$	1
2π	3



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④ $r = f(\theta) = 1 - \sqrt{3}\sin\theta$

Sol.: The polar curve is limacon with inner loop because $a < b$ in $(r = a - b\sin\theta)$ curve.

① Trigonometric function $(\sin\theta) \longrightarrow$

$0 \le \theta \le 2\pi \longrightarrow \left(\theta\text{-step} = \frac{2\pi}{4} = \frac{\pi}{2}\right)$

② ($r = 0$, $r_{max.}$ & $r_{min.}$)

* $r = 0 \longrightarrow 1 - \sqrt{3}\sin\theta = 0 \longrightarrow \sin\theta = 1/\sqrt{3}$

$\therefore \theta = 35.3^\circ (0.2\pi)$ & $144.7^\circ (0.8\pi)$

* $r_{max.} = 1 - \sqrt{3}(-1) = 1 + \sqrt{3} = 2.732$ when

$\sin\theta = -1 \longrightarrow \theta = 3\pi/2$

* $r_{min.} = 1 - \sqrt{3}(1) = -0.732$ when

$\sin\theta = 1 \longrightarrow \theta = \pi/2$

③ Symmetry

$f(-\theta) = 1 - \sqrt{3}\sin(-\theta) = 1 + \sqrt{3}\sin\theta \neq f(\theta)$

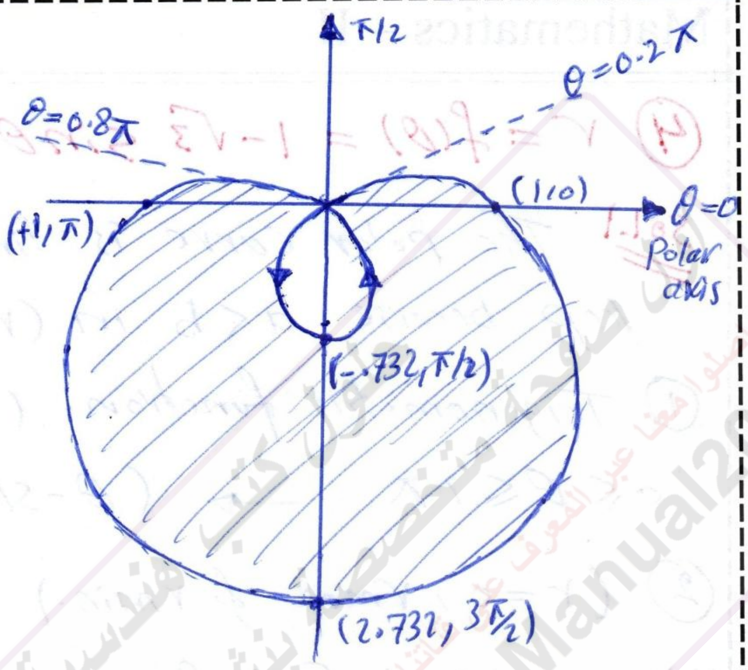
$f(\pi - \theta) = 1 - \sqrt{3}\sin(\pi - \theta) = 1 - \sqrt{3}\sin\theta = f(\theta)$

$\therefore r \neq r \longrightarrow$ The curve is sym. about y-axis only.

* It is clear that the best θ-step is $(\pi/2)$ but should including the θ-values where $(r = 0)$.

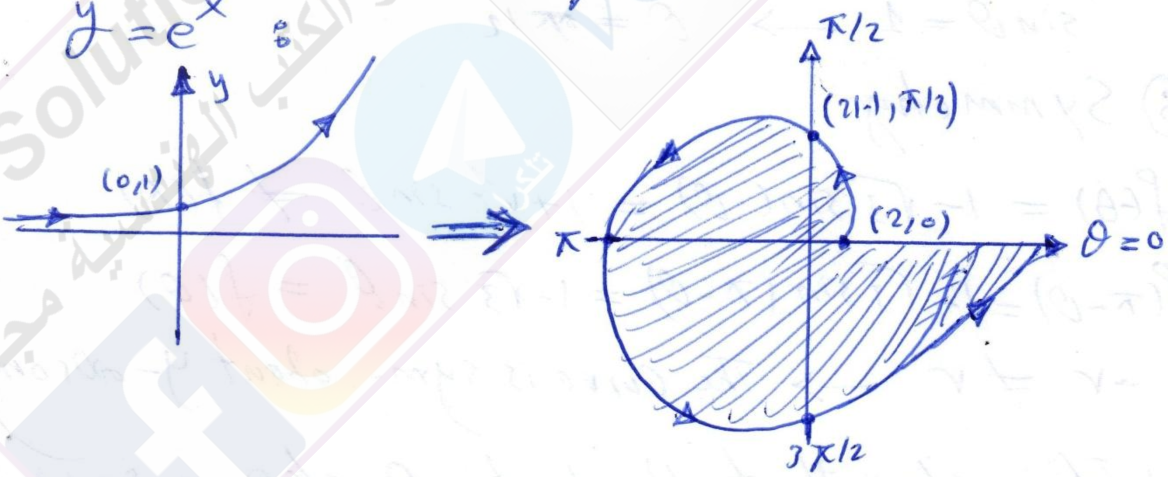
$\longrightarrow$

θ	$r = 1 - \sqrt{3}\sin\theta$
0	1 (lemniscate)
0.2π	0
$\pi/2$	$-.732$
0.8π	0
π	1
$3\pi/2$	2.732
2π	1



⑤ $r = 2e^{3\theta}$ $\quad 0 \le \theta \le 2\pi$

Sol.) The behaviour of this polar logarithmic spiral curve, $(2e^{3\theta})$, is simply concluded from the Cartesian curve of exponential function $y = e^x$:



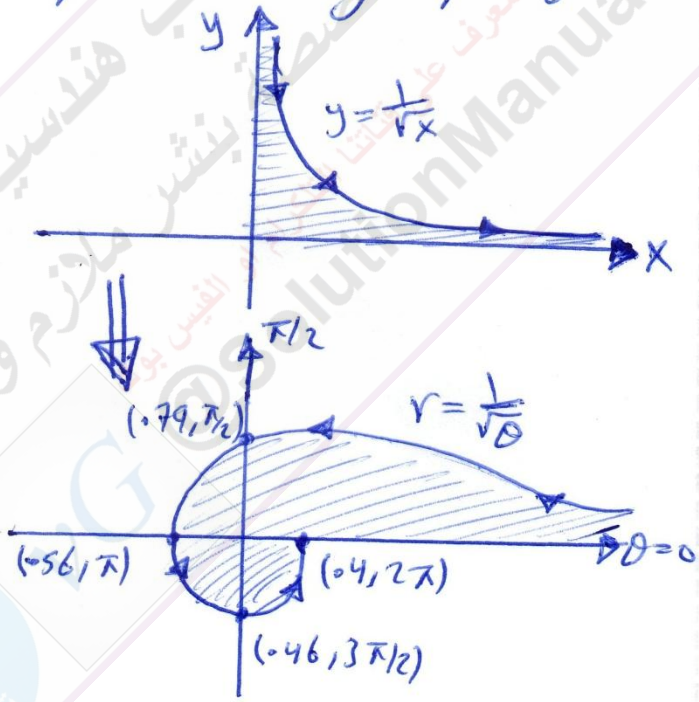
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	2	222	24,783	2,758,821	3.071 e+08

⑥ $r = \frac{1}{\sqrt{\theta}}$ (lituus Spiral Curve), for the first turn only.

Sol.) first turn $\Longleftrightarrow$ $0 < \theta \leq 2\pi$

* As previous example, we are depending on the graph of $r = \frac{1}{\sqrt{\theta}}$ in cartesian (rectangular form) to guess the polar curve graph:

θ	$r = \frac{1}{\sqrt{\theta}}$
0	∞
$\pi/2$	0.79
π	0.56
$3\pi/2$	0.46
2π	0.40



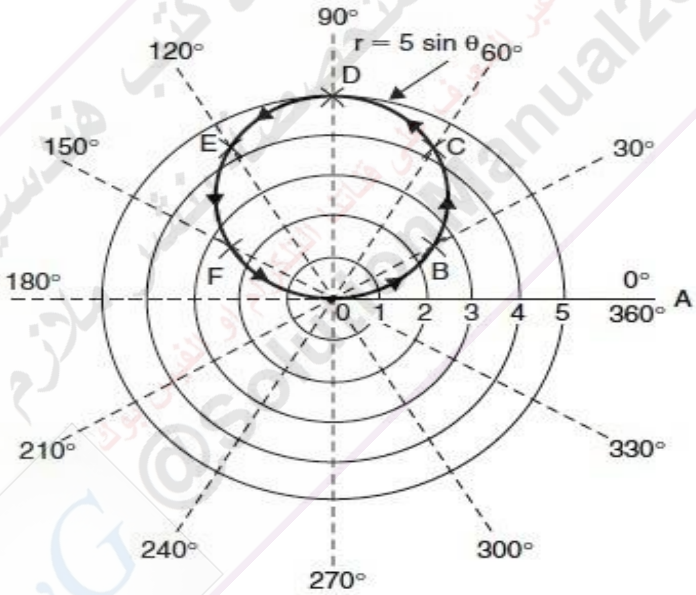
* نلاحظ في هذا المثال وسابقه انه لسنا بحاجة الى اتباع الخطوات الثلاث للتصنيف لفترة الزاوية (θ-step) بسبب عدم وجود تعاظم في منحني الدالة ، بل ان الدالة اما تزايدية increasing مثل ($y = e^x$) او انها تناقصية decreasing ($y = 1/\sqrt{x}$)) .

Application: Plot the polar graph of $r = 5\sin\theta$ between $\theta = 0°$ and $\theta = 360°$ using increments of 30°

A table of values at 30° intervals is produced as shown below.

θ	0	30°	60°	90°	120°	150°	180°
$r = 5\sin\theta$	0	2.50	4.33	5.00	4.33	2.50	0
θ	210°	240°	270°	300°	330°	360°	
$r = 5\sin\theta$	−2.50	−4.33	−5.00	−4.33	−2.50	0	

The graph is plotted as shown in Figure



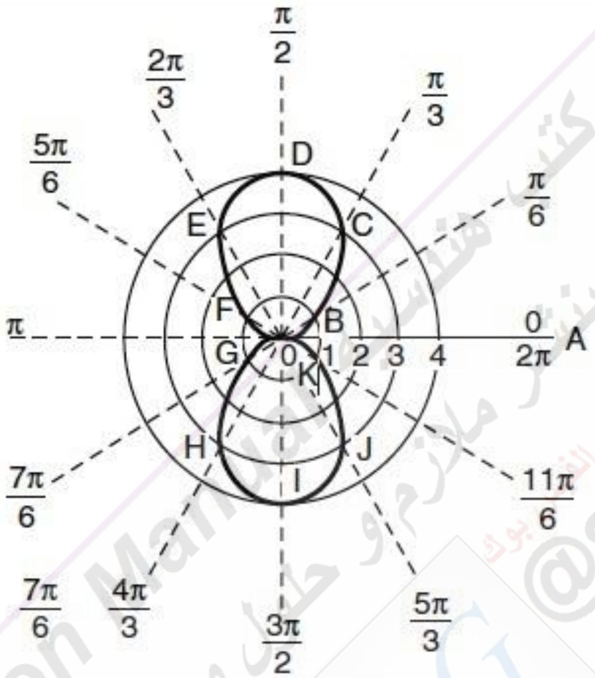
Application: Plot the polar graph of $r = 4\sin^2\theta$ between $\theta = 0$ and $\theta = 2\pi$ radians using intervals of $\frac{\pi}{6}$

A table of values is produced as shown below.

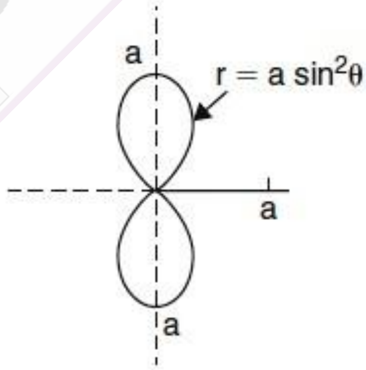
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = 4\sin^2\theta$	0	1	3	4	3	1	0	1	3	4	3	1	0

The zero line 0A is firstly constructed and then the broken lines at intervals of $\frac{\pi}{6}$ rad (or 30°) are produced. The maximum value of r is 4 hence 0A is scaled and circles produced as shown with the largest at a radius of 4 units.

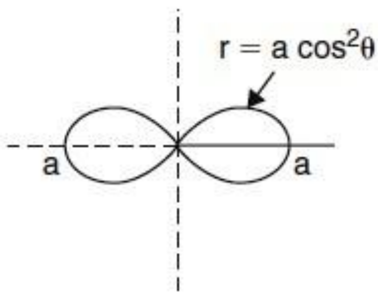
The polar co-ordinates (0, 0), $(1, \frac{\pi}{6})$, $(3, \frac{\pi}{3})$, … $(0, \pi)$ are plotted and shown as points 0, B, C, D, E, F, 0, respectively. Then $(1, \frac{7\pi}{6})$, $(3, \frac{4\pi}{3})$, ··· (0, 0) are plotted as shown by points G, H, I, J, K, 0 respectively. Thus two distinct loops are produced as shown in Figure 6.24.



In general, a polar curve $r = a\sin^2\theta$ is as shown in Figure



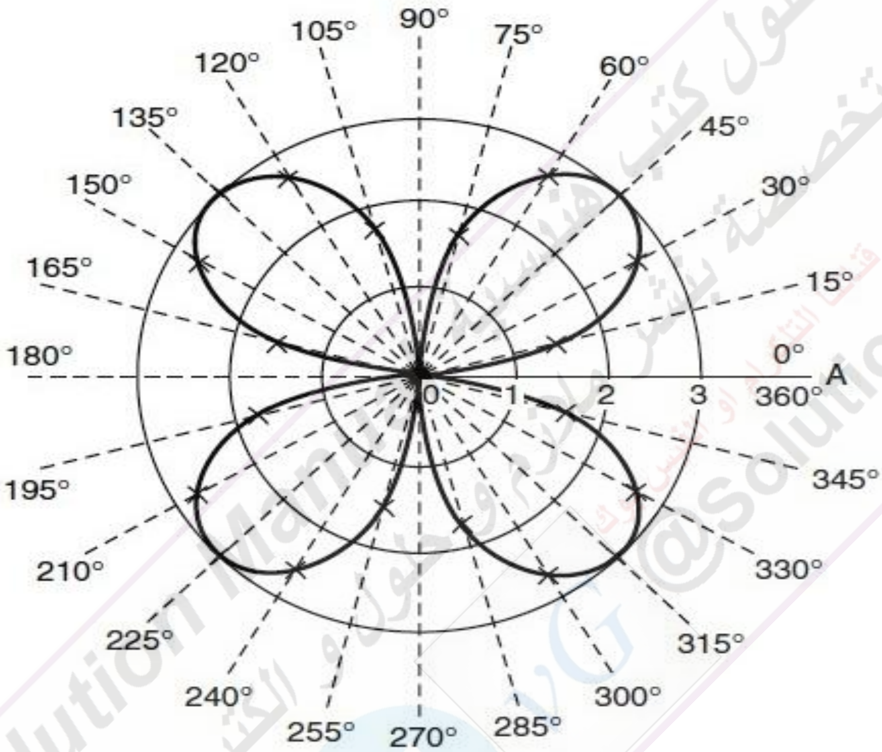
In a similar manner it may be shown that the polar curve $r = a\cos^2\theta$ is as sketched in Figure



Application: Plot the polar graph of $r = 3\sin 2\theta$ between $\theta = 0°$ and $\theta = 360°$, using 15° intervals

As in previous applications a table of values may be produced.

The polar graph $r = 3\sin 2\theta$ is plotted as shown in Figure 6.27 and is seen to contain four similar shaped loops displaced at 90° from each other.



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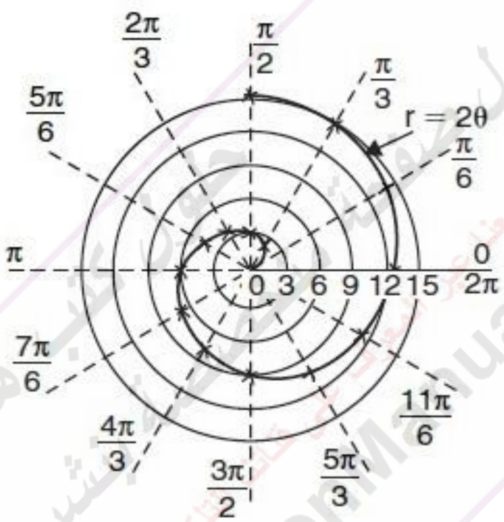
Application: Sketch the polar curve $r = 2\theta$ between $\theta = 0$ and $\theta = \dfrac{5\pi}{2}$ rad at intervals of $\dfrac{\pi}{6}$

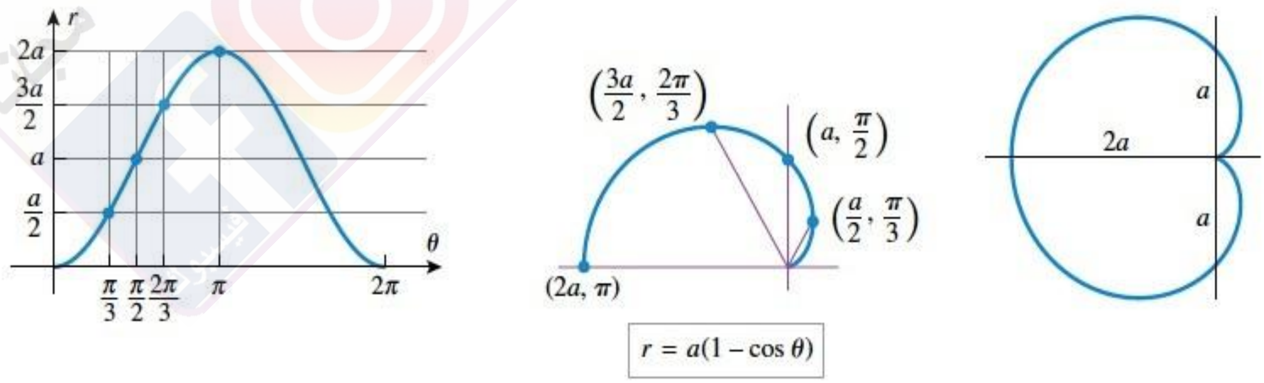
A table of values may be produced and the polar graph of $r = 2\theta$ is shown in Figure and is seen to be an ever-increasing spiral.



▶ **Example 5** Sketch the graph of $r = a(1 - \cos\theta)$ in polar coordinates, assuming a to be a positive constant.

Solution. Observe first that replacing θ by $-\theta$ does not alter the equation, so we know in advance that the graph is symmetric about the polar axis. Thus, if we graph the upper half of the curve, then we can obtain the lower half by reflection about the polar axis.

- As θ varies from 0 to $\pi/3$, r increases from 0 to $a/2$.
- As θ varies from $\pi/3$ to $\pi/2$, r increases from $a/2$ to a.
- As θ varies from $\pi/2$ to $2\pi/3$, r increases from a to $3a/2$.
- As θ varies from $2\pi/3$ to π, r increases from $3a/2$ to $2a$.

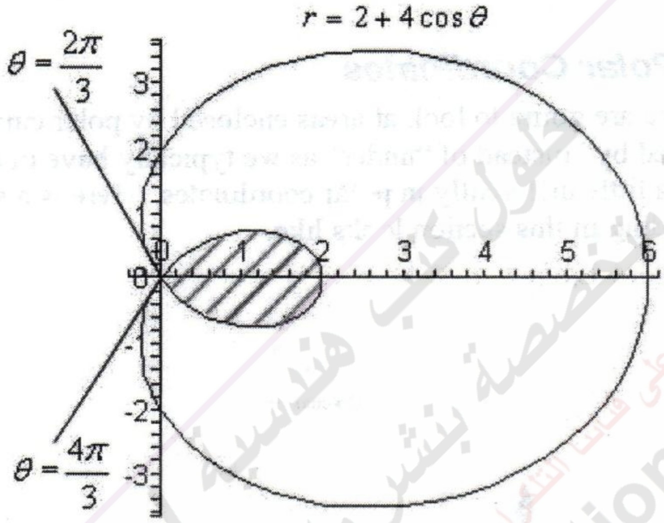


(a) (b) (c)

Example 1 Determine the area of the inner loop of $r = 2 + 4\cos\theta$.

Solution

We graphed this function back when we first started looking at polar coordinates. Here is the sketch again for the sake of completeness.



So, we will need the limits that will enclose this area as we trace out the curve. This will be the angles for which the curve passes through the origin. We can get these by setting the equation equal to zero and solving.

$$0 = 2 + 4\cos\theta$$

$$\cos\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

So, the area is then,

$$A = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(2 + 4\cos\theta\right)^2 d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4 + 16\cos\theta + 16\cos^2\theta\right) d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2 + 8\cos\theta + 4\left(1 + \cos(2\theta)\right) d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6 + 8\cos\theta + 4\cos(2\theta)\, d\theta$$

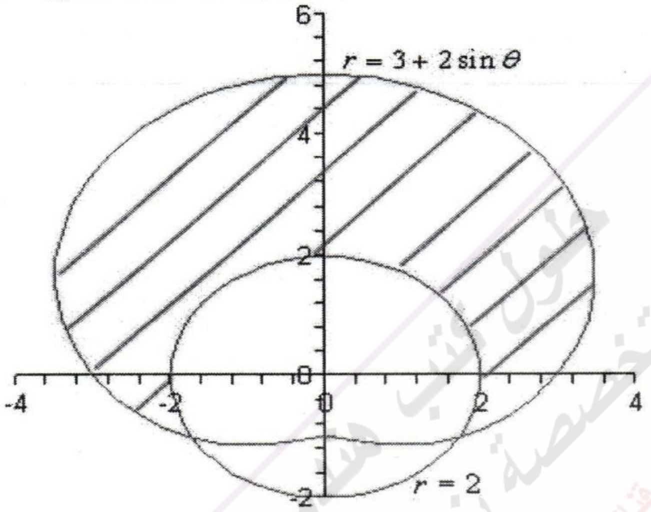
$$= \left(6\theta + 8\sin\theta + 2\sin(2\theta)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= 4\pi - 6\sqrt{3} = 2.174$$

Example 2 Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution

Here is a sketch of the region that we are after.

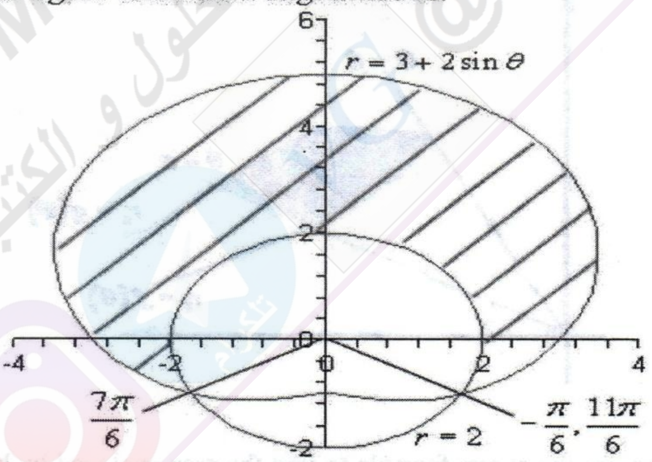


To determine this area we'll need to know that value of θ for which the two curves intersect. We can determine these points by setting the two equations and solving.

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



Note as well here that we also acknowledged that another representation for the angle $\frac{11\pi}{6}$ is $-\frac{\pi}{6}$. This is important for this problem. In order to use the formula above the area must be enclosed as we increase from the smaller to larger angle. So, if we use $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we will not enclose the shaded area, instead we will enclose the bottom most of the three regions. However if we use the angles $-\frac{\pi}{6}$ to $\frac{7\pi}{6}$ we will enclose the area that we're after.

So, the area is then,

$$A=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left((3+2\sin\theta)^2-(2)^2\right)d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left(5+12\sin\theta+4\sin^2\theta\right)d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left(7+12\sin\theta-2\cos(2\theta)\right)d\theta$$

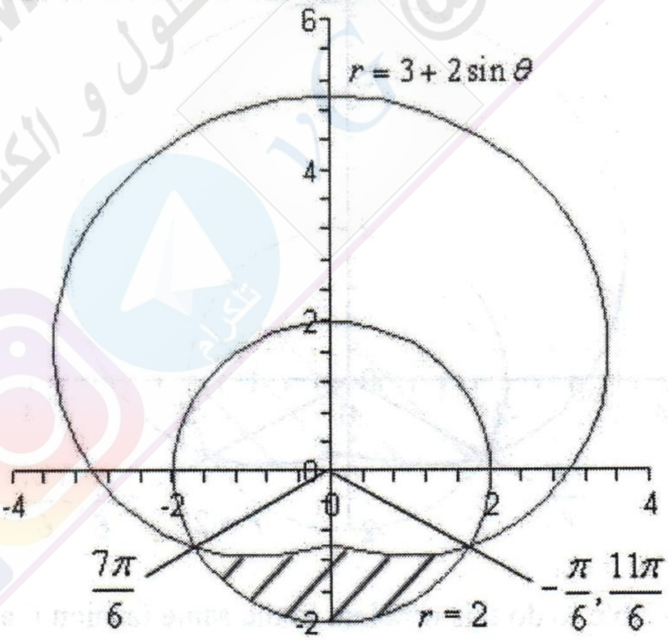
$$=\frac{1}{2}\left(7\theta-12\cos\theta-\sin(2\theta)\right)\Bigg|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$=\frac{11\sqrt{3}}{2}+\frac{14\pi}{3}=24.187$$

Let's work a slight modification of the previous example.

Example 3 Determine the area of the region outside $r=3+2\sin\theta$ and inside $r=2$.

Solution
This time we're looking for the following region.



So, this is the region that we get by using the limits $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$. The area is then,

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left((2)^2 - (3+2\sin\theta)^2\right)d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-5-12\sin\theta-4\sin^2\theta\right)d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-7-12\sin\theta+2\cos(2\theta)\right)d\theta$$

$$= \frac{1}{2}\left(-7\theta+12\cos\theta+\sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

$$= \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} = 2.196$$

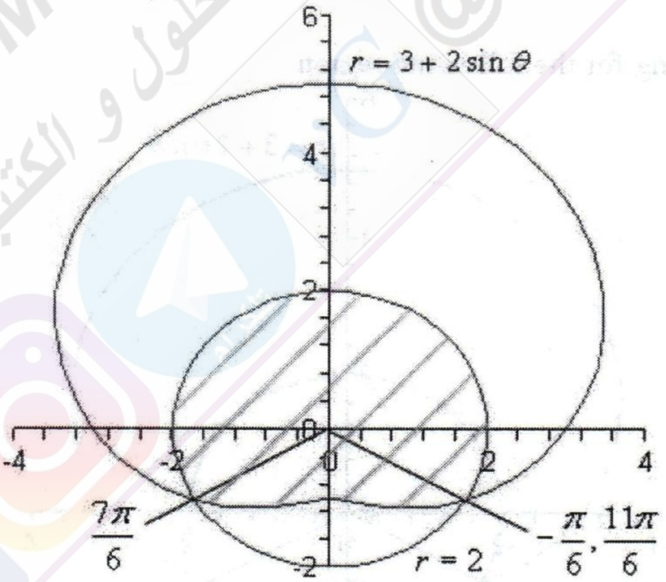
Notice that for this area the "outer" and "inner" function where opposite!

Let's do one final modification of this example.

Example 4 Determine the area that is inside both $r = 3 + 2\sin\theta$ and $r = 2$.

Solution

Here is the sketch for this example.



We are not going to be able to do this problem in the same fashion that we did the previous two. There is no set of limits that will allow us to enclose this area as we increase from one to the other.

In this case however, that is not a major problem. There are two ways to do get the area in this problem.

Solution 1
Notice that the circle is divided up into two portions and we're after the upper portion. Also notice that we found the area of the lower portion in Example 3. Therefore, the area is,

$$\begin{aligned} \text{Area} &= \text{Area of Circle} - \text{Area from Example 3} \\ &= \pi(2)^2 - 2.196 \\ &= 10.370 \end{aligned}$$

Solution 2
In this case we do pretty much the same thing except this time we'll think of the area as the other portion of the limacon than the portion that we were dealing with in Example 2.

So, in this case the area is,

$$\begin{aligned} \text{Area} &= \text{Area of Limacon} - \text{Area from Example 2} \\ &= \int_0^{2\pi} \frac{1}{2}(3+2\sin\theta)^2\, d\theta - 24.187 \\ &= \int_0^{2\pi} \frac{1}{2}\left(9+12\sin\theta+4\sin^2\theta\right) d\theta - 24.187 \\ &= \int_0^{2\pi} \frac{1}{2}\left(11+12\sin\theta+2\cos(2\theta)\right) d\theta - 24.187 \\ &= \frac{1}{2}\left(11\theta-12\cos(\theta)+\sin(2\theta)\right)\Big|_0^{2\pi} - 24.187 \\ &= 11\pi - 24.187 \\ &= 10.370 \end{aligned}$$

A slightly longer approach, but sometimes we are forced to take this longer approach.

Example 1 Determine the length of $r = \theta \quad 0 \le \theta \le 1$.

Solution

Okay, let's just jump straight into the formula since this is a fairly simple function.

$$L = \int_0^1 \sqrt{\theta^2 + 1}\, d\theta$$

We'll need to use a trig substitution here.

$$\theta = \tan x \qquad d\theta = \sec^2 x\, dx$$

$$\theta = 0 \qquad 0 = \tan x \qquad x = 0$$

$$\theta = 1 \qquad 1 = \tan x \qquad x = \frac{\pi}{4}$$

$$\sqrt{\theta^2 + 1} = \sqrt{\tan^2 x + 1} = \sqrt{\sec^2 x} = |\sec x| = \sec x$$

The arc length is then,

$$\begin{aligned} L &= \int_0^1 \sqrt{\theta^2 + 1}\, d\theta \\ &= \int_0^{\frac{\pi}{4}} \sec^3 x\, dx \\ &= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right) \end{aligned}$$

Tangents with Polar Coordinates

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example 1 Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \frac{\pi}{6}$.

Solution

We'll first need the following derivative.

$$\frac{dr}{d\theta} = 8\cos\theta$$

The formula for the derivative $\frac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3+8\sin\theta)\cos\theta}{8\cos^2\theta - (3+8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

Now, at $\theta = \frac{\pi}{6}$ we have $r=7$. We'll need to get the corresponding x-y coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

For the sake of completeness here is a graph of the curve and the tangent line.