(Sequences and Infinite Series)

Sequences (المتتابعات) المتواليات

A sequence is a function whose domain is all integer positive numbers, or simply, an ordered list of numbers.

A sequence can be written as:

$a_1, a_2, a_3, a_4, \ldots$

or denoted by $\{a_n\}_{n=1}^{\infty}$ and simply by $\{a_n\}$.

General Form of Sequences

All terms in a sequence are generated from a general form which relates between the number of the term with its value, this general form called n^{th} term.

Ex) See the following:

① no. of term : 1 2 3 4 5 ---- n
Term : 0 $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{4}{5}$ $\frac{n-1}{n}$

② no. of term : 1 2 3 4 5 ---- n
Term : 0 1 2 3 4 ---- $n-1$

③ no. of term : 1 2 3 4 5 --- n
Term : $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{4}{5}$ $\frac{5}{6}$ --- $\frac{n}{n+1}$

So, $\frac{n-1}{n}$, $n-1$ and $\frac{n}{n+1}$ are called the general form of Sequence (nth term).

Ex) Write first five terms of the following sequences :

① $\frac{n+2}{n^2}$, ② $\frac{n+1}{e^n}$ and ③ $\frac{1}{n^2} \sin \frac{\pi}{n}$

Sol.)

① $a_n = \frac{1+2}{(1)^2}, \frac{2+2}{(2)^2}, \frac{3+2}{(3)^2}, \frac{4+2}{(4)^2}, \frac{5+2}{(5)^2}$

$= 3, \frac{4}{4}, \frac{5}{9}, \frac{6}{16}, \frac{7}{25}$

② $a_n = \frac{2}{e^1}, \frac{3}{e^2}, \frac{4}{e^3}, \frac{5}{e^4}, \frac{6}{e^5}$

③ $a_n = \sin\pi, \frac{1}{4}\sin(\pi/2), \frac{1}{9}\sin(\pi/3), 1/16 \sin(\pi/4)$

$1/25 \sin(\pi/5)$

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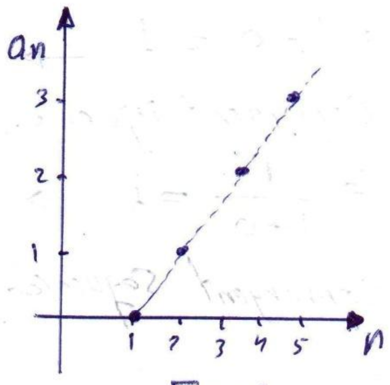
Convergence and Divergence of Sequences

When sequences approach limiting values as (n) increases, then these sequences called " Convergent " while the other sequences called " Divergent ".

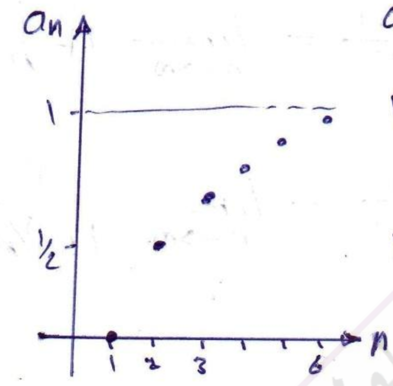
So, we can investigate the behaviour of sequence by plotting it in n-a_n plane.

Ex) Study the behaviour of following sequences : ① $a_n = n-1$ ② $a_n = \frac{n-1}{n}$ & ③ $a_n = \frac{n}{n+1}$

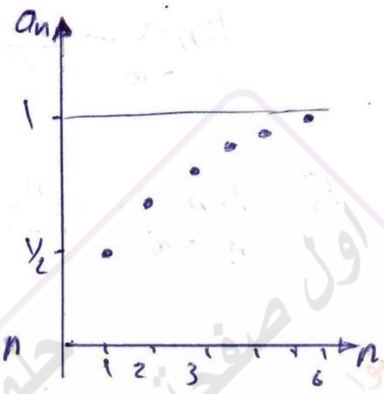
Sol.) ① $a_n = n-1$, when value of n increase, the value of a_n increase too with no limiting value, so the sequence is (Divergent). See Fig. →



Fig(1)



Fig(2)



Fig(3)

② $a_n = \frac{n-1}{n}$,

the value of a_n will approach a limiting value (1) when the value of n increases. So, the sequence is convergent.

③ $a_n = \frac{n}{n+1}$

This sequence have the same behaviour of above one, when n increase a_n will approach (convergent) to (1).

Def. If a sequence $\{a_n\}$ converges, then:

$\lim_{n\to\infty} a_n = L$, and call L the limit of the sequence. If no such limit exists, then the sequence $\{a_n\}$ is divergent.

Ex) Test the following sequences and find the limit of the convergent sequences:

① $\{n-1\}$ ② $\{\frac{n-1}{n}\}$, ③ $\{\frac{n}{n+1}\}$

Sol.) ① $\lim_{n\to\infty} (n-1) = \infty$ → no limit exist → divergent sequence

② $\lim\limits_{n\to\infty} \frac{n-1}{n} = \lim\limits_{n\to\infty} 1-\frac{1}{n} = 1-0 = 1 \longrightarrow$ Convergent sequence.

③ $\lim\limits_{n\to\infty} \frac{n}{n+1} = \lim\limits_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{1+0} = 1 \longrightarrow$ Convergent sequence.

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Infinite Series المتسلسلات غير المنتهية

A sequence of the form:

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

is called an infinite Series. The term a_n called the nth term of the Series.

The sequence $\{S_n\}$ defined by:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2 = S_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3 = S_2 + a_2$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the Sequence of partial sums of the Series and the number S_n being the nth partial sum.

If the sequence of partial sum S_n converges to a limiting value (L), then the series is **convergent** and its sum is (L). Hence,

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

If S_n of series does not converge, then the Series **divergent** and there is no sum.

* Any Series can be expressed by:

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} a_k\right) \text{ or Simply } \sum a_n \left(\sum a_k\right)$$

The following examples shows how the "partial fraction technique" can be used to compute the sums for some series. When this sum can be done, then the series is convergent.

Ex) Determine whether the following series is convergent or no? ① $\sum \frac{2}{n(n+1)}$

Sol.) $\sum \frac{2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow A(n+1) + Bn = 2 \rightarrow$

if $n=-1 \rightarrow B=-2$

$n=0 \rightarrow A=2 \quad \rightarrow$

$\sum \frac{2}{n(n+1)} = \sum \left(\frac{2}{n} - \frac{2}{n+1}\right) \rightarrow$

$S_n = \underbrace{(2-\cancel{1})}_{n=1} + \underbrace{(\cancel{1}-\cancel{\tfrac{2}{3}})}_{n=2} + \underbrace{(\cancel{\tfrac{2}{3}}-\cancel{\tfrac{2}{4}})}_{n=3} + \cdots + \underbrace{(\cancel{\tfrac{2}{n-1}}-\cancel{\tfrac{2}{n}})}_{n=n-1} + \underbrace{(\cancel{\tfrac{2}{n}}-\tfrac{2}{n+1})}_{n=n}$

$\therefore S_n = 2 - \frac{2}{n+1}$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(2 - \frac{2}{n+1}\right) = 2 \rightarrow \therefore$ the series is convergent

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Ex) Discuss the convergence of this series: $\sum \frac{4}{(4n-3)(4n+1)}$?

Sol.) $\frac{4}{(4n-3)(4n+1)} = \frac{A}{4n-3} + \frac{B}{4n+1} \Rightarrow A(4n+1) + B(4n-3) = 4$

if $n = 3/4 \rightarrow A=1$ and if $n=-1/4 \rightarrow B=-1 \rightarrow$

$\therefore \sum \frac{4}{(4n-3)(4n+1)} = \sum \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)$

$S_n = \underbrace{(1-\cancel{\tfrac{1}{5}})}_{n=1} + \underbrace{(\cancel{\tfrac{1}{5}}-\cancel{\tfrac{1}{9}})}_{n=2} + \underbrace{(\cancel{\tfrac{1}{9}}-\cancel{\tfrac{1}{13}})}_{n=3} + \cdots + \left(\cancel{\tfrac{1}{4n-3}} - \tfrac{1}{4n+1}\right)$

$= 1 - \frac{1}{4n+1} \longrightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{4n+1}\right) = 1 \rightarrow$

$\therefore$ the series is convergent

Power Series

- An expression of the form :

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \quad \text{—— (1a)}$$

is a power series centered at $x = 0$.

- An expression of the form :

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n + \cdots \quad \text{(1b)}$$

is a power series centered at $x = a$. The term $a_n(x-a)^n$ is the <u>nth term</u>; the number <u>a</u> is the center.

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Power Series can be used for representing the functions whose possess a derivatives of all orders at the center (zero for eq. 1a and a for eq. 1b) and continuous on its domains only.

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Ex) Check whether the following functions can be represented by Power series : $\sin x$, $\cos x$, e^{ax}, $\ln x$ & $\sqrt{x}$?

Sol.) (1) $f(x) = \sin x \rightarrow f'(x) = \cos x \rightarrow f'(0) = 1$ ✓

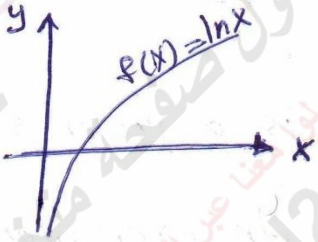
(2) $f(x) = \cos x \rightarrow f'(x) = -\sin x \rightarrow f'(0) = 0$ ✓

(3) $f(x) = e^{ax} \rightarrow f'(x) = a e^{ax} \rightarrow f'(0) = a$ ✓

Then the above functions can be represented by Power Series.

(4) $f(x) = \ln x \rightarrow f'(x) = \frac{1}{x} \rightarrow f'(0) = \frac{1}{0} = \nexists$

the function $\ln x$ can not be represented by power series because neither $f(x)$ nor its derivative $f'(x)$ exist at $x = 0$.



(5) $f(x) = \sqrt{x} \rightarrow f'(x) = \frac{1}{2\sqrt{x}} \rightarrow f'(0) = \frac{1}{0} = \nexists$

the function $\sqrt{x}$ can not be represented by power series since $f(x) = \sqrt{x}$ have no derivative @ $x = 0$.

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Maclaurin Series (Power Series about $x = 0$)

Let $f(x)$ be a function with derivatives of all orders at $x = 0$ are exists.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \quad \text{— (1)}$$

then by successive diffrentiations, we got:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$f''(x) = 2a_2 + (3\times 2)a_3 x + (4\times 3)a_4 x^2 + \cdots$$

$$f'''(x) = (3\times 2)a_3 + (4\times 3\times 2)a_4 x + \cdots$$

$$\vdots \qquad \vdots$$

Putting $x = 0$ in each term above, yields:

$f(0) = a_0$

$f'(0) = a_1 \longrightarrow a_1 = f'(0)/1!$

$f''(0) = 2a_2 \longrightarrow a_2 = f''(0)/2!$

$f'''(0) = 3\times2\, a_3 \longrightarrow a_3 = f'''(0)/3!$

$\vdots \qquad \vdots \qquad \vdots$

$f^{n}(0) = ------ n!\,x_n \longrightarrow a_n = f^{n}(0)/n!$

by sub. the coefficients $(a_1, a_2, \ldots, a_n)$ in eq. ① , got:

$$\left[f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{n}(0) + \cdots \right] \quad ②$$

the above eq. is a power series about $(x=0)$ and known as Mac laurin Series for function $f(x)$.

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Ex) Find the Maclaurin's Series for the function $f(x) = \sin(mx)$?

Sol) $\because f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{n}(0)$ ——②

then;

$f(x) = \sin mx \longrightarrow f(0) = 0$

$f'(x) = m\cos mx \longrightarrow f'(0) = m$

$f''(x) = -m^2 \sin mx \longrightarrow f''(0) = 0$

$f'''(x) = -m^3 \cos mx \longrightarrow f'''(0) = -m^3$

$f^{4}(x) = m^4 \sin mx \longrightarrow f''''(0) = 0$

$f^{5}(x) = m^5 \cos mx \longrightarrow f'''''(0) = m^5$

by sub. in ② above $\longrightarrow$ get $\therefore (f(x) = \sin mx) \rightarrow$

$$\sin mx = 0 + xm + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-m^3) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}m^5 + \cdots$$

$$\sin mx = mx - m^3\frac{x^3}{3!} + m^5\frac{x^5}{5!} - m^7\frac{x^7}{7!} + m^9\frac{x^9}{9!} - \cdots$$

Now putting $(m=1) \rightarrow$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

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EX) Find Maclaurin's Series for $f(x) = \ln(1+x)$?

Sol.) $\therefore f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$f(x) = \ln(1+x) \rightarrow f(0) = \ln(1+0) = 0$

$f'(x) = 1/(1+x) \rightarrow f'(0) = 1$

$f''(x) = (-1)/(1+x)^2 \rightarrow f''(0) = -1$

$f'''(x) = (2)/(1+x)^3 \rightarrow f'''(0) = 2$

$f^{iv}(x) = (-6)/(1+x)^4 \rightarrow f^{iv}(0) = -6 \rightarrow$

$$\ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots \rightarrow$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}}{n} x^n + \cdots$$

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Remark

We can above replacing each x by $-x$ to get:

$$\ln(1-x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \cdots$$

Check signs !

Ex) Find Maclaurin's Series for $f(x)=e^{ax}$, then compute (e^2) ?

Sol.)

$$f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f''''(0)+\cdots+\frac{x^n}{n!}f^{n}(0)+\cdots$$

$f(x)=e^{ax} \longrightarrow f(0)=1$

$f'(x)=a e^{ax} \longrightarrow f'(0)=a$

$f''(x)=a^2 e^{ax} \longrightarrow f''(0)=a^2$

$f'''(x)=a^3 e^{ax} \longrightarrow f'''(0)=a^3$

$\vdots \qquad \vdots$

$f^{n}(x)=a^n e^{ax} \longrightarrow f^{n}(0)=a^n \longrightarrow$

$$e^{ax}=1+ax+a^2\frac{x^2}{2!}+a^3\frac{x^3}{3!}+\cdots+a^n\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{a^n x^n}{n!}$$

Putting $a=1 \longrightarrow$

$$e^{x}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$\longrightarrow$ Putting $x=2 \longrightarrow$ for example take up $(n=6)$

$$e^2=1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\frac{2^5}{5!}+\frac{2^6}{6!}=7.35555$$

e^2 (calculator) $=7.389$

H.W Find the Maclaurin's Series for the following functions :

① $f(x) = \cos x \longrightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

② $f(x) = (1+x)^3 \longrightarrow (1+x)^3 = 1 + 3x + \frac{6x^2}{2!} + \frac{6x^3}{3!}$

③ $f(x) = \tan x \longrightarrow \tan x = x + \frac{x^3}{3} + \frac{2x^5}{12} + \cdots$

④ $f(x) = \sinh x \longrightarrow \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

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Taylor's Series (Power Series about $x=a$)

Let $f(x)$ be a continuous function with derivatives of all orders are exists at $(x=a)$, then the Taylor series generated by $f(x)$ @ $x=a$ is :

$$\boxed{\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots} \quad \text{③}$$

It's clear that, like in Maclaurin's series, in Taylor's series $f(x)$ can be represented as follows :

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n + \cdots \quad \text{④}$$

Now to find the coefficients $(a_0, a_1, \cdots, a_n)$, we

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follow the same procedure like in Maclaurin's series (from eq.1 to eq.2) but with replacing $(x=0)$ by $(x=a)$, got the Taylor polynomial of order n about $x=a$:

$$\left[f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots \right] \quad \text{— (5)}$$

$**\sim\sim**$

Ex) Find Taylor's series for $f(x) = \ln x$ about $x=1$?

Sol.) $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$ (eq-5)

if $a=1 \rightarrow$

$$f(1) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots$$

$f(x) = \ln x \longrightarrow f(1) = \ln 1 = 0$

$f'(x) = 1/x \longrightarrow f'(1) = 1$

$f''(x) = -(1/x^2) \longrightarrow f''(1) = -1$

$f'''(x) = (2/x^3) \longrightarrow f'''(1) = 2$

$f^{IV}(x) = (-6/x^4) \longrightarrow f^{IV}(1) = -6 \rightarrow$

$$\ln x = 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$*\sim*$

Ex) Find Taylor's series for $f(x)=\sqrt{x}$ about $x=4$?

Sol.) using eq. 5 $\rightarrow$

$$f(x)=f(4)+(x-4)f'(4)+\frac{(x-4)^2}{2!}f''(4)+\frac{(x-4)^3}{3!}f'''(4)+\cdots$$

hence,

$f(x)=\sqrt{x} \rightarrow f(4)=2$

$f'(x)=1/2\sqrt{x} \rightarrow f'(4)=1/4$

$f''(x)=-1/4x^{3/2} \rightarrow f''(4)=-1/32$

$f'''(x)=3/8x^{5/2} \rightarrow f'''(4)=3/256 \rightarrow$

$$f(x)=\sqrt{x}=2+(x-4)\left(\frac{1}{4}\right)+\frac{(x-4)^2}{2!}\left(-\frac{1}{32}\right)+\frac{(x-4)^3}{3!}\left(\frac{3}{256}\right)+\cdots$$

$$\therefore \sqrt{x}=2+\frac{(x-4)}{4}-\frac{(x-4)^2}{64}+\frac{(x-4)^3}{512}+\cdots$$

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Ex) Find Taylor series for $f(x)=\sin x$ about $x=\frac{\pi}{6}$?

Sol.) $f(x)=f\left(\frac{\pi}{6}\right)+\left(x-\frac{\pi}{6}\right)f'\left(\frac{\pi}{6}\right)+\frac{(x-\pi/6)^2}{2!}f''\left(\frac{\pi}{6}\right)+\cdots$

$f(x)=\sin x \rightarrow f(\pi/6)=\sin\pi/6=\frac{1}{2}$

$f'(x)=\cos x \rightarrow f'(\pi/6)=\sqrt{3}/2$

$f''(x)=-\sin x \rightarrow f''(\pi/6)=-1/2$

$f'''(x)=-\cos x \rightarrow f'''(\pi/6)=-\sqrt{3}/2 \rightarrow$ by Sub. in eq. above

$$\sin x=\frac{1}{2}+\left(x-\frac{\pi}{6}\right)\frac{\sqrt{3}}{2}+\frac{\left(x-\frac{\pi}{6}\right)^2}{2!}\left(-\frac{1}{2}\right)+\frac{\left(x-\frac{\pi}{6}\right)^3}{3!}\left(-\frac{\sqrt{3}}{2}\right)+\cdots$$

$$=\frac{1}{2}+\left(x-\frac{\pi}{6}\right)\frac{\sqrt{3}}{2}-\frac{(x-\pi/6)^2}{2*2!}-\frac{\sqrt{3}}{2*3!}\left(x-\frac{\pi}{6}\right)^3+\cdots$$

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Ex) Find the Maclaurin's Series for $f(x) = e^{-x}$ about $x=0$.

Sol.) Depending on the example solved in Page (6):

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

then by replacing each (x) in above series with $(-x)$:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3} + \cdots$$

وهناك طريقة أخرى للحل وهي الطريقة التقليدية كما في مثال صفحة 6.

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Ex) Find the Power Series for $f(x) = x^4 e^{-3x^2}$ about $x=0$.

Sol.) since: Maclaurian's Series of e^x is:

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Longrightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\therefore e^{-3x^2} = \sum_{n=0}^{\infty} \frac{(-3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{n!}$$

$$\Longrightarrow x^4 e^{-3x^2} = \sum_{n=0}^{\infty} \frac{(-3)^n x^4 x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+4}}{n!}$$

حيث تم حل اعلاه بكتابة متسلسلة e^{x} بالطريقة التقليدية سابقة ومنه
ثم ايجاد المتسلسلة المطلوبة بالطريقة السريعة اعلاه.

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Ex) Find the Power Series for $f(x) = \cos(x)$ about $x=0$.

Sol.) Power Series about $x=0 \Longleftrightarrow$ Maclaurian's Series.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{n}(0)$$

$\Longrightarrow$

$f^{(0)}(x) = \cos x \longrightarrow f^{0}(0) = 1$

$f'(x) = -\sin x \longrightarrow f'(0) = 0$

$f''(x) = -\cos x \longrightarrow f''(0) = -1$

$f'''(x) = \sin x \longrightarrow f'''(0) = 0$

$f^{(4)}(x) = \cos x \longrightarrow f^{(4)}(0) = 1$

$\vdots \qquad \vdots$

$$\therefore f(x) = \cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \cdots$$

$$= \underbrace{1}_{n=0} + \underbrace{0}_{n=1} - \underbrace{\frac{1}{2!} x^2}_{n=2} + \underbrace{0}_{n=3} + \underbrace{\frac{1}{4!} x^4}_{n=4} + \underbrace{0}_{n=5} - \underbrace{\frac{1}{6!} x^6}_{n=6} + \cdots$$

$$\Longrightarrow \cos x = \underbrace{1}_{n=0} - \underbrace{\frac{1}{2!} x^2}_{n=1} + \underbrace{\frac{1}{4!} x^4}_{n=2} - \underbrace{\frac{1}{6!} x^6}_{n=3} + \cdots$$

then after renumbering the terms, the general formula became:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Applications of Series

a) The Term-by-Term Differentiation Theorem

Ex) Find the $f'(x)$ for the following Power Series:

$$f(x) = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$(-1 \le x \le 1)$.

Sol.)

$\therefore \; f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

by using the term by term differentiation theorem for the above original Power Series; we get

$$f'(x) = 1 - \frac{3x^2}{3} + \frac{5x^4}{5} - \frac{7x^6}{7} + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

b) The Term-by-Term Integration of Power Series

Ex) If a function series of a polynomial form is

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots \quad [\text{Converge at } -1 < t < 1]$$

Find the integration of this series.

Sol.) using term by term integration technique:

$$\int_0^x \frac{1}{1+t}\,dt = \ln(1+x) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} \Big]_0^x$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 \le x \le 1$$

Ex) Evaluate the following integral about $x=0$

$$\int \frac{\sin x}{x} \quad \text{using Power Series technique.}$$

Sol.) At first we should find the Polynomial representation, using Power-Series, for the $\frac{\sin x}{x}$.
This is easy; at first we should find the Power series for $\sin x$ about $x=0$, then divide the result series by x.

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(See P.P.5)

$$\Rightarrow \frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

then simply;

$$\int \frac{\sin x}{x}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\,dx$$

$$= \left[C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \right]$$

which is called the Series representation for the integral.

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