



Class: 2st

Subject: Mathematics

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Equation of a line in Space :-

If L is a line in space that passes through a point $P_0(x_0, y_0, z_0)$ and it parallel to a vector $V = Ai + Bj + Ck$.

Then $P(x, y, z)$ is any point lies on L only if

$$\vec{P_0P} = tV \quad \text{--- (1)}$$

where t is some number.

So eq(1) can be write:

$$\therefore (x-x_0)i + (y-y_0)j + (z-z_0)k = t(Ai + Bj + Ck)$$

$$\underline{x - x_0 = At} \quad ; \quad \underline{y - y_0 = Bt} \quad ; \quad \underline{z - z_0 = Ct}$$

$$\therefore \underline{x = At + x_0} \quad ; \quad \underline{y = Bt + y_0} \quad ; \quad \underline{z = Ct + z_0}$$

EX: Find parametric equations for the line through the point $(-2, 0, 4)$ parallel to the vector $V = 2i + 4j - 2k$

Solution:

$$P_0(x_0, y_0, z_0) = (-2, 0, 4)$$

$$Ai + Bj + Ck = 2i + 4j - 2k$$

$$\underline{x = 2t - 2}$$

$$\underline{y = 4t}$$

$$\underline{z = -2t + 4 = 4 - 2t}$$



Ex: Find parametric equations for line through the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$?

Solution:

$$\vec{PQ} = (1 - (-3))i + ((-1) - 2)j + (4 - (-3))k$$

$$= 4i - 3j + 7k$$

with $(x_0, y_0, z_0) = (-3, 2, -3)$

$$x = 4t - 3$$

$$y = -3t + 2$$

$$z = 7t - 3$$

and for $(x_0, y_0, z_0) = (1, -1, 4)$

$$x = 4t + 1$$

$$y = -3t - 1$$

$$z = 7t + 4$$

The distance from a point to a line:

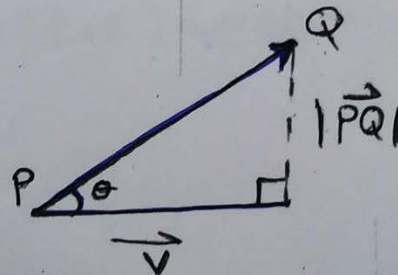
I- The first method:

- ① Find the point $Q(x, y, z)$.
- ② Calculate the distance from P to Q .

II- The second method:

The distance from a point Q to a line that passes through a point P parallel to a vector

$$d = \frac{|\vec{PQ} \times \vec{V}|}{|\vec{V}|}$$



III- The third method:

$$d = |\vec{PQ}| * |\sin \theta|$$



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Ex: Find the distance from point $P(1, 1, 5)$ to the line $x=1+t$; $y=3-t$; $z=2t$

Solution: -

I- First method

$$Q(x, y, z) = (1+t, 3-t, 2t)$$

$$\vec{PQ} = (1+t-1)\mathbf{i} + (3-t-1)\mathbf{j} + (2t-5)\mathbf{k} = t\mathbf{i} + (2-t)\mathbf{j} + (2t-5)\mathbf{k}$$

$$|\vec{PQ}| = \sqrt{(t)^2 + (2-t)^2 + (2t-5)^2}$$

$$= \sqrt{t^2 + 4 - 4t + t^2 + 4t^2 - 20t + 25}$$

$$= \sqrt{6t^2 - 24t + 29} \quad \text{بالترتيب}$$

$$\text{let } (|\vec{PQ}|)^2 = f(t) = d^2$$

$$f(t) = 6t^2 - 24t + 29$$

$$\frac{df}{dt} = 0 = 12t - 24 \Rightarrow \boxed{t=2}$$

نعوض قيمة $(t=2)$ في معادلة $(|\vec{PQ}|)$

$$|\vec{PQ}| = \sqrt{6*(2^2) - (24*2) + 29} = \sqrt{5} = d$$

نعوض قيمة $(t=2)$ في النقطة (Q)

$$Q = (1+2, 3-2, 2*2)$$

$$\boxed{Q = (3, 1, 4)}$$



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II Second method :-

$$P(1, 1, 5)$$

$$\text{line: } x=1+t, \quad y=3-t, \quad z=2t$$

$$\vec{v} = i - j + 2k$$

$$Q = (1, 3, 0)$$

$$\vec{PQ} = (1-1)i + (3-1)j + (0-5)k$$

$$= 2j - 5k$$

$$\vec{PQ} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & 2 & -5 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= (2 \times 2 - (-1 \times -5))i - (0 \times 2 - 1 \times -5)j + (0 \times -1 - 1 \times 2)k$$

$$= -i - 5j - 2k$$

$$d = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{(-1)^2 + (-5)^2 + (-2)^2}}{\sqrt{(1)^2 + (-1)^2 + (2)^2}}$$

$$= \frac{\sqrt{30}}{\sqrt{6}}$$

$$= \boxed{\sqrt{5}}$$



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III - Third method :-

$$P = (1, 1, 5)$$

$$\text{line: } X = 1 + t, \quad y = 3 - t, \quad z = 2t$$

From eqs of line ($X = 1 + t, y = 3 - t, z = 2t$)

$$\vec{V} = i - j + 2k \quad \& \quad Q = (1, 3, 0)$$

$$\vec{PQ} = 0i + 2j - 5k$$

$$\cos \theta = \frac{\vec{PQ} \cdot \vec{V}}{|\vec{PQ}| \cdot |\vec{V}|}$$

$$= \frac{(0 * 1) + (2 * -1) + (-5 * 2)}{\sqrt{0^2 + 2^2 + (-5)^2} * \sqrt{1^2 + (-1)^2 + 2^2}} = \frac{-12}{\sqrt{29} * \sqrt{6}}$$

$$\cos \theta = -0.9097 \Rightarrow \theta = \cos^{-1}(-0.9097)$$

$$\theta = 155.5^\circ$$

$$d = |\vec{PQ}| * \sin \theta$$

$$= \sqrt{29} * \sin(155.5^\circ)$$

$$= 2.2332 = \sqrt{5}$$



Equation of the plane :-

To find the equation of the plane that passes through the point $P_0(x_0, y_0, z_0)$ and its normal vector is

$$\vec{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Let $P(x, y, z)$ be any point in the plane

$$\vec{P_0P} = (x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}$$

$$\vec{P_0P} \perp \vec{N} \Rightarrow \vec{P_0P} \cdot \vec{N} = 0$$

$$\Rightarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

$$\boxed{ax + by + cz = d} \iff \text{Equation of the Plane}$$

Ex:- Find the equation of the plane having the points $A(2, 3, 5)$; $B(7, 2, 1)$ and $C(1, 1, 1)$

Solution:-

$$\vec{AB} = (7-2)\mathbf{i} + (2-3)\mathbf{j} + (1-5)\mathbf{k} \\ = \boxed{5\mathbf{i} + \mathbf{j} - 4\mathbf{k}}$$

$$\vec{AC} = (1-2)\mathbf{i} + (1-3)\mathbf{j} + (1-5)\mathbf{k} = \boxed{-\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}}$$

$$\vec{N} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & -4 \\ -1 & -2 & -4 \end{vmatrix} = \boxed{-4\mathbf{i} + 24\mathbf{j} - 11\mathbf{k}}$$

$$\therefore -4x + 24y - 11z = d \quad \begin{array}{l} \text{نوضم أحد النقاط (A, B, C)} \\ \text{في المعادلة لإيجاد قيمة (d)} \end{array}$$

$$C(1, 1, 1) \Rightarrow (-4 * 1) + (24 * 1) - (11 * 1) = d \Rightarrow \boxed{d = -9}$$

$$\therefore \boxed{-4x + 24y - 11z = -9} \text{ is the eq of the plane.}$$

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The distance from a point to a plane:-

If P is a point with normal \vec{N} , then the distance from any point (Q) to the plane is the length of the vector projection of \vec{PQ} onto \vec{N}

$$d = \left| \text{proj}_{\vec{N}} \vec{PQ} \right| = \left| \frac{\vec{PQ} \cdot \vec{N}}{|\vec{N}|} \right|$$

Ex:- Find the distance from the point $P(3, 1, 3)$ to the plane whose equation is $3x - 5y + z = 4$.

Solution:-

Find any point in the plane assume $A(0, 0, 4)$

$$\vec{AP} = (3-0)\mathbf{i} + (1-0)\mathbf{j} + (3-4)\mathbf{k} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\vec{N} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$$

$$d = \left| \text{proj}_{\vec{N}} \vec{AP} \right| = \left| \frac{\vec{AP} \cdot \vec{N}}{|\vec{N}|} \right|$$

$$= \frac{|(3*3) + (1*-5) + (-1*1)|}{\sqrt{(3)^2 + (-5)^2 + (1)^2}}$$

$$= \frac{3}{\sqrt{35}}$$



Distance between two lines :-

Note: let A be a point on line L_1

B be a point on line L_2

- ① If $\vec{L}_1 \times \vec{L}_2 = \vec{0}$, then the two lines are parallel and the distance is defined by

$$d = |\vec{AB}| \times |\sin \theta| \quad \text{where } \theta = \cos^{-1} \left(\frac{|\vec{AB} \cdot \vec{L}_1|}{|\vec{AB}| \cdot |\vec{L}_1|} \right)$$

- ② If $\vec{L}_1 \times \vec{L}_2 \neq \vec{0}$ and $\vec{AB} \cdot (\vec{L}_1 \times \vec{L}_2) \neq 0$, then the two lines are skew-lines and the distance is defined by

$$d = \left| \text{Proj}_{\vec{N}} \vec{AB} \right| \quad \text{where } \vec{N} = \vec{L}_1 \times \vec{L}_2$$

- ③ If $\vec{L}_1 \times \vec{L}_2 \neq \vec{0}$ and $\vec{AB} \cdot (\vec{L}_1 \times \vec{L}_2) = 0$ then the two lines are intersecting and the distance between them is Zero.

$$d = 0$$

Ex: - Find the distance between the two lines

$$\textcircled{1} \quad \frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-8}{1} \quad ; \quad \frac{x}{3} = \frac{y-2}{5} = \frac{z-8}{-8}$$

Solution: -

$$\left. \begin{array}{l} \vec{L}_1 = i - 2j + k; A = (1, 1, 8) \\ \vec{L}_2 = 3i + 5j - 8k; B = (0, 2, 8) \end{array} \right\} \Rightarrow \vec{AB} = -i + j$$

$$\vec{AB} \cdot (\vec{L}_1 \times \vec{L}_2) = \begin{vmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 3 & 5 & -8 \end{vmatrix} = -1(-2 \cdot 8 - 5 \cdot 1) - 1(1 \cdot 8 - 3 \cdot 1) + 0(1 \cdot 5 - 3 \cdot 2) \\ = -11 + 11 + 0 = 0$$

$\therefore d = 0$ the two lines intersect



Ex:- Find the distance between the two lines

$$(2) \frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{3} ; \frac{x-2}{10} = \frac{y+2}{5} = \frac{z-3}{15}$$

Solution:-

$$\vec{L}_1 = 2\vec{i} + \vec{j} + 3\vec{k} ; A = (1, 2, -3) \quad \left. \vphantom{\vec{L}_1} \right\} \Rightarrow \vec{AB} = \vec{i} - 4\vec{j} + 6\vec{k}$$

$$\vec{L}_2 = 10\vec{i} + 5\vec{j} + 15\vec{k} ; B = (2, -2, 3)$$

$$\vec{L}_1 \times \vec{L}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ 10 & 5 & 15 \end{vmatrix} = (1 \cdot 15 - 5 \cdot 3)\vec{i} - (2 \cdot 15 - 10 \cdot 3)\vec{j} + (2 \cdot 5 - 1 \cdot 10)\vec{k}$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k} =$$

$$\therefore \vec{L}_1 \parallel \vec{L}_2$$

$$\cos \theta = \frac{\vec{AB} \cdot \vec{L}_1}{|\vec{AB}| |\vec{L}_1|} = \frac{(1 \cdot 2) + (-4 \cdot 1) + (6 \cdot 3)}{\sqrt{(1)^2 + (-4)^2 + (6)^2} \sqrt{(2)^2 + (1)^2 + (3)^2}} = \frac{16}{\sqrt{53} \sqrt{14}}$$

$$\theta = 54.03^\circ$$

$$d = |\vec{AB}| \cdot |\sin \theta| = \sqrt{53} |\sin 54.03| \approx \boxed{5.89} \text{ units}$$

$$(3) \frac{x-1}{2} = \frac{y-1}{2} = \frac{z+3}{3} ; \frac{x-2}{1} = \frac{y+2}{-1} = \frac{z-3}{7}$$

Solution:-

$$\vec{L}_1 = 2\vec{i} + 2\vec{j} + 3\vec{k} ; A = (1, 1, -3) \quad \left. \vphantom{\vec{L}_1} \right\} \Rightarrow \vec{AB} = \vec{i} - 3\vec{j} + 6\vec{k}$$

$$\vec{L}_2 = \vec{i} - \vec{j} + 7\vec{k} ; B = (2, -2, 3)$$

$$\vec{N} = \vec{L}_1 \times \vec{L}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 3 \\ 1 & -1 & 7 \end{vmatrix} = 17\vec{i} - 11\vec{j} - 4\vec{k}$$

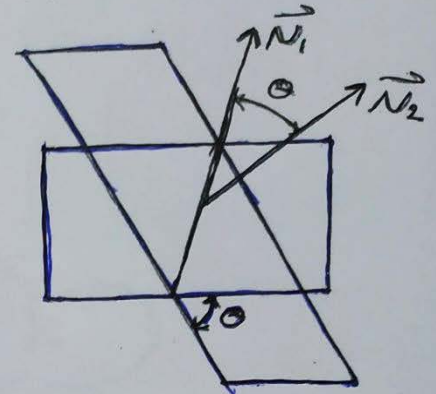
$$d = \left| \text{Proj}_{\vec{N}} \vec{AB} \right| = \frac{|\vec{AB} \cdot \vec{N}|}{|\vec{N}|} = \frac{(1 \cdot 17) + (-3 \cdot -11) + (6 \cdot -4)}{\sqrt{(17)^2 + (-11)^2 + (-4)^2}}$$

$$= \frac{26}{\sqrt{426}} = \boxed{1.25} \text{ units}$$

Angles between planes :-

The angle between two intersecting planes is defined to be the angle determined by their normal vectors

$$\theta = \cos^{-1} \left(\frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1| |\vec{N}_2|} \right)$$



Ex:- Find the angle between the planes

$$3x - 6y - 2z = 15 \quad ; \quad 2x + y - 2z = 5$$

Solution:-

$$\vec{N}_1 = 3i - 6j - 2k \quad ; \quad \vec{N}_2 = 2i + j - 2k$$

$$\theta = \cos^{-1} \left(\frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1| |\vec{N}_2|} \right)$$

$$= \cos^{-1} \left(\frac{(3*2) + (-6*1) + (-2*-2)}{\sqrt{(3)^2 + (-6)^2 + (-2)^2} \cdot \sqrt{(2)^2 + (1)^2 + (-2)^2}} \right)$$

$$\therefore \theta = \cos^{-1} \left(\frac{4}{21} \right) = \boxed{79^\circ}$$



Vector functions :-

$$\vec{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$$

$$* \mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$\mathbf{R}(t)$: is the position vector

$$* \mathbf{v}(t) = \frac{d\mathbf{R}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$\mathbf{v}(t)$: is the velocity vector

$$* \mathbf{a}(t) = \frac{d^2\mathbf{R}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

$\mathbf{a}(t)$: is the acceleration vector

$$* |\mathbf{v}| = \sqrt{\left(\frac{d\mathbf{R}}{dt}\right)^2} = \text{Speed}$$

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{R}/dt}{\sqrt{(d\mathbf{R}/dt)^2}} = \text{Direction}$$

Ex :- The vector $\mathbf{R}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$ gives the position of a moving body at time. Find the body's speed and direction when $t=2$.

Solution :-

$$\mathbf{v}(t) = \frac{d\mathbf{R}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k}$$

$$\text{Speed} = |\mathbf{v}(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2}$$

$$\text{at } t=2 \Rightarrow |\mathbf{v}(2)| = \sqrt{(-3\sin 2)^2 + (3\cos 2)^2 + (2 \times 2)^2} = \boxed{5}$$

$$\text{Direction} = \frac{\mathbf{v}(2)}{|\mathbf{v}(2)|} = \frac{1}{5}(-3\sin 2\mathbf{i} + 3\cos 2\mathbf{j} + 4\mathbf{k})$$



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H.W No (2)

① Find the acute angles between the line

(a) $3x + y = 5$; $2x - y = 4$

(b) $12x + 5y = 1$; $2x - 2y = 3$

②

a. Find the area of the triangle determined by $P(1, 1, 1)$

$Q(2, 1, 3)$ and $R(3, -1, 1)$

b. Find a unit vector perpendicular to plane PQR

③ Let $u = 5i - j + k$; $v = j - 5k$ and $w = -15i + 3j - 3k$
which vectors are (a) perpendicular (b) parallel ?

④ Find the point where the line
 $x = \frac{8}{3} + 2t$, $y = -2t$ and $z = 1 + t$
intersects the plane $3x + 2y + 6z = 6$

⑤ Find the distance from $S(1, 1, 3)$ to the plane
 $3x + 2y + 6z = 6$

⑥ Find the parametric equations for the lines for
a/ The line through the point $P(3, 4, 1)$ parallel to the
vector $i + j + k$

b/ The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$

c/ The line through the origin parallel to vector $2j + k$

d/ The line through the point $(3, -2, 1)$ parallel to the
line $x = 1 + 2t$, $y = 2 - t$, $z = 3t$



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- 7- Find equations for the planes for
- The plane through $P_0(0, 2, -1)$ normal to $\vec{N} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
 - The plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$
 - The plane through $P_0(2, 4, 5)$ perpendicular to the line $x = 5 + t$; $y = 1 + 3t$; $z = 4t$
 - The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A
- 8- Find the distance from the point $S(2, 1, 3)$ and the line $x = 2 + 2t$, $y = 1 + 6t$ and $z = 3$
- 9- Find the distance from the point $S(2, 2, 3)$ to the plane $2x + y + 2z = 4$
- 10- Find the angle between the planes $5x + y - z = 10$; $x - 2y + 3z = -1$
- 11- Find the acute angle between planes $4y + 3z = 12$; $3x + 2y + 6z = 6$