Matrix:.

Matrix calculus is a mathematical toll used in connection with linear equations , linear transformations , systems of differential equations etc. matrices are important in physics, engineering, statistics etc.

Matrix is an array of numbers. A matrix with **m** rows and **n** columns is order **m** x **n** and is shown as follows..

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ & & \cdot & \\ & & a_{ij} & \\ & & \cdot & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

1-Matrices addition and Subtraction:.

If two matrices **A** and **B** can be added or subtracted if and only if their dimensions are the same (i.e . both matrices have the same numder of rows and columns).

a-Addition:.

If **A** and **B** above are matrices of the same type then the sum is found by adding the corresponding elements $a_{ij} + b_{ij}$.

Example 1:. If A $= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$,B = $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$, find A+ B.

Solution:. A+B = $\begin{bmatrix} 1+2 & 2+1 & 3+2 \\ 1+1 & 0+0 & 2+3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 5 \\ 2 & 0 & 5 \end{bmatrix}$.

b-Subtraction :.

If **A** and **B** above are matrices of the same type then the subtraction is found by subtracting the corresponding elements $a_{ij} - b_{ij}$.

Example 2:. If A $= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$,B = $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$, find A- B.

Solution:. A - B = $\begin{bmatrix} 1-2 & 2-1 & 3-2 \\ 1-1 & 0-0 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

Example 3:. If A = $\begin{bmatrix} -3 & 0 \\ 7 & -4 \end{bmatrix}$,B = $\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$ and C = $\begin{bmatrix} -1 & 0 \\ -2 & -4 \end{bmatrix}$,find A- B+C.

Solution:. A – B+C = $\begin{bmatrix} -3-2-1 & 0+1+0 \\ 7+7-2 & -4-4-4 \end{bmatrix} = \begin{bmatrix} -6 & 1 \\ 12 & -12 \end{bmatrix}$.

2- Scalar multiplication:.

We multiply (or divide)each element by the scalar value (a single number).

Example 4:. If A= $\begin{bmatrix} 3 & 1 \\ 7 & -1 \\ 2 & 8 \end{bmatrix}$, find (3A),($\frac{1}{2}$ A).

Solution:.

$$3A = 3\begin{bmatrix} 3 & 1 \\ 7 & -1 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 3\,x\,3 & 3\,x\,1 \\ 3\,x\,7 & 3\,x-1 \\ 3\,x\,2 & 3\,x\,8 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 21 & -3 \\ 6 & 24 \end{bmatrix}.$$

$$\frac{1}{2}A = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 7 & -1 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{7}{2} & \frac{-1}{2} \\ 1 & 4 \end{bmatrix}$$

$$\frac{1}{2}A = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 7 & -1 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{7}{2} & \frac{-1}{2} \\ \frac{2}{2} & \frac{8}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{7}{2} & \frac{-1}{2} \\ 1 & 4 \end{bmatrix}.$$

3-Matrix multiplication:.

When the number of columns of the first matrix is the same as the number of rows in the second matrix then matrix multiplication can performed. If $\mathbf{A_{ij}}$ **,** $\mathbf{B_{jk}}$ then ($\mathbf{A_{ij} \times B_{jk} = C_{ik}}$)

-Here is an example of matrix multiplication for two 2x2 matrices.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (ae+bg) & (af+bh) \\ (ce+dg) & (cf+dh) \end{bmatrix}.$$

-Here is an example of matrix multiplication for two 3x3 matrices.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} =$$

$$\begin{bmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+en+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+ir) \end{bmatrix}.$$

Example5:. If A $= \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$,B = $\begin{bmatrix} -5 & 7 \\ -3 & 4 \end{bmatrix}$, Find(A×B).

Solution:. A×B = $\begin{bmatrix} 2*-5 + 3*-3 & 2*7 + 3*4 \\ 1*-5 + -4*-3 & 1*7 + -4*4 \end{bmatrix}$

$= \begin{bmatrix} -19 & 26 \\ 7 & -9 \end{bmatrix}$

Example 6:.If A= $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ B=$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}$,Find(A×B).

Solution:. A×B=

$$\begin{bmatrix} 1*2+4*4+5*1 & 1*3+4*2+5*3 & 1*1+4*3+5*1 \\ 2*2+1*4+3*1 & 2*3+1*2+3*3 & 2*1+1*3+3*1 \\ 1*2+2*4+4*1 & 1*3+2*2+4*3 & 1*1+2*3+4*1 \end{bmatrix}.$$

$$=\begin{bmatrix} 23 & 26 & 18 \\ 11 & 17 & 8 \\ 14 & 19 & 11 \end{bmatrix}.$$

Example7:. If A= $\begin{bmatrix} 1 & -1 \\ 4 & 2 \\ 0 & 3 \end{bmatrix}$,B=$\begin{bmatrix} 1 & 5 & 6 & 7 \\ -2 & 4 & 8 & -4 \end{bmatrix}$, Find(A×B).

Solution:.

A×B =

$$\begin{bmatrix} 1*1+ -1*-2 & 1*5+ -1*4 & 1*6+ -1*8 & 1*7+-1*-4 \\ 4*1+2*-2 & 4*5+2*4 & 4*6+2*8 & 4*7+2*-4 \\ 0*1+3*-2 & 0*5+3*4 & 0*6+3*8 & 0*7+3*-4 \end{bmatrix}$$

$$=\begin{bmatrix} 3 & 1 & -2 & 11 \\ 0 & 28 & 40 & 20 \\ -6 & 12 & 24 & -12 \end{bmatrix}$$

H.W:- Q 1. The matrices A to K are:

$A=\begin{bmatrix} 3 & -1 \\ -4 & 7 \end{bmatrix}$, $B=\begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{-1}{3} & \frac{-3}{5} \end{bmatrix}$, $C=\begin{bmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{bmatrix}$

$D=\begin{bmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{bmatrix}$, $E=\begin{bmatrix} 3 & 6 & \frac{1}{2} \\ 5 & \frac{-2}{3} & 7 \\ -1 & 0 & \frac{5}{3} \end{bmatrix}$, $F=\begin{bmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{bmatrix}$

$G=\begin{bmatrix} \frac{3}{4} \\ \frac{7}{5} \end{bmatrix}$, $H=\begin{bmatrix} -2 \\ 5 \end{bmatrix}$, $J=\begin{bmatrix} 4 \\ -11 \\ 7 \end{bmatrix}$, $K=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Determine:-

1. A+B $\begin{bmatrix} 3\frac{1}{2} & -\frac{1}{3} \\ -4\frac{1}{3} & 6\frac{2}{5} \end{bmatrix}$ 2. D+E $\begin{bmatrix} 7 & -1 & 6\frac{1}{2} \\ 3 & 3\frac{1}{3} & 7 \\ 4 & 7 & -2\frac{1}{3} \end{bmatrix}$

3. A-B $\begin{bmatrix} 2\frac{1}{2} & -1\frac{2}{3} \\ -3\frac{2}{3} & 7\frac{3}{5} \end{bmatrix}$ 4. A+B-C $\begin{bmatrix} 4.8 & -7.73 \\ -6.83 & 10.3 \end{bmatrix}$

5. 5A+6B $\begin{bmatrix} 18.0 & -1.0 \\ -22.0 & 31.4 \end{bmatrix}$ 6. 2D+3E-4F $\begin{bmatrix} 4.6 & -5.6 & -12.1 \\ 17.4 & -9.2 & 28.6 \\ -14.2 & 0.4 & 16.2-4.8 \end{bmatrix}$

7. A×H $\begin{bmatrix} -11 \\ 43 \end{bmatrix}$ 8. A×B $\begin{bmatrix} 1\frac{5}{6} & 2\frac{3}{5} \\ -4\frac{1}{3} & -6\frac{13}{15} \end{bmatrix}$

9. A×C $\begin{bmatrix} -6.4 & 26.1 \\ 22.7 & -56.9 \end{bmatrix}$ 10. D×J $\begin{bmatrix} 135 \\ -52 \\ -85 \end{bmatrix}$

11. E×K $\begin{bmatrix} 3\frac{1}{2} & 6 \\ 12 & -\frac{2}{3} \\ \frac{2}{3} & 0 \end{bmatrix}$ 12. D×F $\begin{bmatrix} 55.4 & 3.4 & 10.1 \\ -12.6 & 10.4 & -20.4 \\ -16.9 & 25.0 & 37.9 \end{bmatrix}$

13. Show that A×C $\neq$ C×A

{ A×C $\begin{bmatrix} -6.4 & 26.1 \\ 22.7 & -56.9 \end{bmatrix}$, C×A $\begin{bmatrix} -33.5 & -53.1 \\ 23.1 & -29.8 \end{bmatrix}$ }

Q 2 . Can multiply $A=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 3 & 7 \\ 1 & 2 & 5 \end{bmatrix}$

Determinants :-

Determinants are arrays which are very useful in the analysis and solution of systems of linear algebraic equations.

- ***The determinant of a 2×2 matrix*** A = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Det(A) or |A| is defined to be determinant

|A| = a*d – b*c

- ***The determinant of a 3×3 matrix*** A = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$|A| = c_{11} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + c_{12} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + c_{13} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$c_{ij} = (-1)^{i+j}$

$|A| = a_{11}(a_{22}*a_{33} – a_{23}*a_{32}) – a_{12} (a_{21} *a_{33}) + a_{13}(a_{21}*a_{32} – a_{22}*a_{31})$

Example 8:. Find determinants for the following:

1- $\begin{vmatrix} 3 & -2 \\ 7 & 4 \end{vmatrix}$ = (3*4) - (-2*7) = 12- (-14) = 26

2- $\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$

= 1(-2) – 5 (0) + 0 (-2) = -2

3- $\begin{vmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{vmatrix} = 4\begin{vmatrix} 4 & 0 \\ 7 & -4 \end{vmatrix} - (-7) \begin{vmatrix} -2 & 0 \\ 5 & -4 \end{vmatrix} + 6 \begin{vmatrix} -2 & 4 \\ 5 & 7 \end{vmatrix}$

= 4(-16-0) +7 (8-0) + 6 (-14-20) = -212

4- $\begin{vmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{-1}{3} & \frac{-3}{5} \end{vmatrix} = \frac{1}{2} * \frac{-3}{5} - \left(\frac{2}{3} * \frac{-1}{3}\right) = \frac{-3}{10} + \frac{2}{9} = \frac{-7}{90}$

Notes:.

The value of det. Of any matrix is the same if the expansion is done by using any row or column.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Example 9:. $\begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 2 & -1 & 1 \end{vmatrix}$

- If we take the first row.

$= 1\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = 5+10-5= 10$

- If we take the second row.

$= -(-1)\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} -2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 3-3+10= 10$

- If we take the first column.

$= 1\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 5+3+2= 10$

- ***The determinant of a 4×4 matrix*** $A = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$

$= 0\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} -1\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= -1(0\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}) = -1$

H.W:. Evaluate the following determinants:.

1. $\begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 3 \end{vmatrix}$ ans. (-5)

2. $\begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{vmatrix}$ ans. (0)

3. $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix}$ ans. (-7) [by used the third row]

4. $\begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix}$ ans. (6) [by used the first column]

5. $\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 7 \\ 3 & 0 & 2 & 1 \end{vmatrix}$ ans. (38)

6. $\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & 2 & 6 \\ 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & -5 \end{vmatrix}$ ans. (2)

Solution of simultaneous equations by using Cramers rule

Definition: Crammer ' s rule uses a method of determinants to solve systems of equations.

Two simultaneous equations in x and y

ax+by=p

cx+dy=q

to solve use the following

then $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$

where:

$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $D_x = \begin{vmatrix} p & b \\ q & d \end{vmatrix}$, $D_y = \begin{vmatrix} a & p \\ c & q \end{vmatrix}$

Example 10:. Solve the system by Crammer's rule .

3x-y=9

x+2y=-4

solution:

$D = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 6+1= 7$

$D_x = \begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix} = 18-4= 14$, $D_y = \begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix} = -12-9 = -21$

$x = \frac{14}{7} = 2$, $y = \frac{-21}{7} = -3$

Three simultaneous equations in x , y and z

$a_{11}x + a_{12}y + a_{13}z = b_1$
$a_{21}x + a_{22}y + a_{23}z = b_2$
$a_{31}x + a_{32}y + a_{33}z = b_3$

To solve use the following

then $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$

where

$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $D_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$

$D_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$, $D_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

Example 11:. Solve the system by using Crammer's rule .

$$2x + y + 3z = 9$$
$$x - 2y + z = -2$$
$$3x + 2y + 2z = 7$$

Solution:

$$D = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 2 & 2 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 13$$

$$D_x = \begin{vmatrix} 9 & 1 & 3 \\ -2 & -2 & 1 \\ 7 & 2 & 2 \end{vmatrix} = 9\begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ 7 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & -2 \\ 7 & 2 \end{vmatrix} = -13$$

$$D_y = \begin{vmatrix} 2 & 9 & 3 \\ 1 & -2 & 1 \\ 3 & 7 & 2 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 7 & 2 \end{vmatrix} - 9\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 \\ 3 & 7 \end{vmatrix} = 26$$

$$D_z = \begin{vmatrix} 2 & 1 & 9 \\ 1 & -2 & -2 \\ 3 & 2 & 7 \end{vmatrix} = 2\begin{vmatrix} -2 & -2 \\ 2 & 7 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 3 & 7 \end{vmatrix} + 9\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 39$$

then $\boldsymbol{x = \frac{-13}{13} = -1}$, $\boldsymbol{y = \frac{26}{13} = 2}$, $\boldsymbol{z = \frac{39}{13} = 3}$

W.H:. Solve the system by using Crammer's rule .

1. $x - 3y = 6$
   $2x + 3y = 3$ ans. $x = 3$, $y = -1$

2. $x + 2y + 3z = 5$
   $2x - 3y - z = 3$
   $-3x + 4y + 5z = 3$ ans. $x = 1$, $y = -1$, $z = 2$

3. $3a + 4b - 3c = 2$
   $-2a + 2b + 2c = 15$
   $7a - 5b + 4c = 26$ ans. $x = 2.5$, $y = 3.5$, $z = 6.5$

4. $2x - 4y - 6 = 0$
   $x + y = 1 - z$
   $5y + 7z = 10$ ans. $x = 0$, $y = -1.5$, $z = 2.5$

5. $x_1 + x_2 + x_3 + x_4 = 3$
   $2x_1 - 2x_2 - x_3 + 2x_4 = 0$
   $3x_1 - x_2 + 2x_3 + 2x_4 = 2$
   $x_1 - x_2 - 2x_3 + x_4 = 0$ ans. $x_1 = 1/2$, $x_2 = 3/2$, $x_3 = 0$, $x_4 = 1$