Laplace Transform

Convert Differential Equations $\rightarrow$ Algebric Equations

Very important to analyze the control system.

If $f(t)$ is a continuous function, then $f(t)$ is defined by using the Laplace transform

$$\mathcal{L}\{f(t)\} \quad \text{or} \quad F(s)$$

Laplace Transform Formula

$$\boxed{F(s) = \int_0^\infty f(t) \cdot e^{-s \cdot t} \cdot dt}$$

where:

t : a real variable to be converted to a complex function

$t \geq 0$

Ex/ Compute $\mathcal{L}\{1\}$

sol/ $f(t) = 1$

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st} \, dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^\infty = \left[\frac{e^{-\infty}}{-s} - \frac{e^0}{-s} \right]$$

$$= \left[\frac{1}{s \cdot e^{\infty}} - \frac{e^0}{s} \right] = \left[0 - \frac{1}{-s} \right] = \boxed{\frac{1}{s}}$$

Ex2/ Compute $\mathcal{L}\{e^{at}\}$

Sol/
$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} \cdot e^{-st} \cdot dt = \int_0^\infty e^{(a-s)t} \, dt$$

$$= \left[\frac{e^{(a-s)t}}{(a-s)} \right]_0^\infty = \left[\frac{e^{\infty}}{a-s} - \frac{e^{0}}{a-s} \right]$$

$$= \left(0 - \frac{1}{a-s} \right) = \boxed{\frac{1}{s-a}}$$

Laplace Transform Table

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$		
1	$\frac{1}{s}$		
e^{at}	$\frac{1}{(s-a)}$		
t^n	$\frac{n!}{s^{(n+1)}}$		
$\sin(at)$	$\frac{a}{s^2+a^2}$	$e^{at} \cdot \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$e^{at} \cdot \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$e^{at} \cdot \sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$e^{at} \cdot \cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
$t^n \, e^{at}$	$\frac{n!}{(s-a)^{(n+1)}}$		

properties of the Laplace Transform $f(t) = 0$ for $t < 0$ 3

property	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
Multiplication by constant	$K\ f(t)$	$k\ F(s)$
Linearity	$K_1\ f_1(t) + K_2\ f_2(t)$	$K_1\ F_1(s) + K_2\ F_2(s)$
Time scaling	$f(at)\ ,\ a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-T)\ u(t-T)$	$e^{-Ts}\ F(s)$
Frequency shift	$e^{-at}\ f(t)$	$F(s+a)$
Time 1^{st} derivative	$f' = \frac{df}{dt}$	$s\ F(s) - f(0^-)$
Time 2^{nd} derivative	$f'' = \frac{d^2 f}{dt^2}$	$s^2\ F(s) - s\ f(0)$
Time integral	$\int_0^t f(\tau)\, d\tau$	$\frac{1}{s} F(s)$
Frequency derivative	$t\ f(t)$	$-\frac{d}{ds} F(s)$
Frequency integral	$\frac{f(t)}{t}$	$\int_0^\infty F(s')\, ds'$

Ex 3/ Using the Laplace transform find the solution for the following equation

$$\frac{\partial y(t)}{\partial x} = 3 - 2t$$

with initial conditions

$$y(0) = 0$$

Derivative $\longrightarrow$ $D\, y(0) = 0$

Sol

$Y(s) = \mathcal{L}\, y(t)$, Perform Laplace transform for both sides

From Laplace transform table

$$s\, Y(s) - y(0) = \frac{3}{s} - 2\,\frac{1!}{s^{1+1}}$$

$$= \frac{3}{s} - \frac{2}{s^2} \longrightarrow \boxed{\begin{array}{c} \left(\frac{3}{s} \times \frac{s}{s}\right) - \left(\frac{2}{s^2}\right) \\ \frac{3s}{s^2} - \frac{2}{s^2} \\ \frac{3s-2}{s^2} \end{array}}$$

$$\therefore \quad Y(s) = \frac{y(0)\, s^2 + 3s - 2}{s^3}$$

$$= \frac{y(0)\, s^2}{s^3} + \frac{3s}{s^3} - \frac{2}{s^3}$$

$$= y(0)\,\frac{1}{s} + 3\,\frac{1}{s^2} - \frac{2}{s^3}$$

$$Y(s) = y(0) + 3\, t - t^2$$

initial condition $y(0) = 0$

$$\boxed{Y(s) = 3t - t^2}$$ General solution

without the Laplace transform

$$\boxed{y(t) = -t^2 + 3t}$$

EX4/ Using the Laplace transform find the Solution for the following equation

$$\frac{\partial y(t)}{\partial t} = e^{-3t}$$

with initial conditions

$$y(0) = 4$$

$$D y(0) = 0$$

Sol/

$$Y(s) = \mathcal{L}\, y(t)$$

$$\mathcal{L}^{-1}(Y(s)) = y(t)$$

perform Laplace for both sides

$$s\,Y(s) - y(0) = \frac{1}{s-(-3)} = \frac{1}{s+3}$$

$$s\,Y(s) = \frac{1}{s+3} + y(0) \longrightarrow \frac{1}{s+3} + y(0) * \frac{s+3}{s+3}$$

$$\frac{1 + y(0)s + y(0)3}{s+3}$$

$$Y(s) = \frac{1 + y(0)s + 3\,y(0)}{s\,(s+3)}$$

Use Inverse Laplace transform

$$Y(s) = \frac{y(0)\,(s+3) + 1}{s\,(s+3)}$$

$$= y(0)\,\frac{1}{s} + \frac{1}{s(s+3)}$$

partial fraction كسور جزئية

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{(s+3)}$$

نضرب المعادلة بالمقام

$$1 = A(s+3) + Bs$$

If $s = 0$, If $s = -3$

$$\boxed{A = \frac{1}{3}} \qquad \boxed{B = \frac{-1}{3}}$$

نعوض

$$\frac{\frac{1}{3}}{s} + \frac{\frac{-1}{3}}{s+3}$$

$$\boxed{\frac{1}{3s} - \frac{1}{3(s+3)}}$$

$$= \mathcal{L}^{-1}\, y(0)\frac{1}{s} + \mathcal{L}^{-1}\left(\frac{1}{3s}\right) - \mathcal{L}^{-1}\left(\frac{1}{3(s+3)}\right)$$

$$Y(s) = y(0) + \frac{1}{3} - \frac{1}{3}e^{-3t}$$

initial conditions

$$Y(s) = \frac{13}{3} - \frac{1}{3}e^{-3t}$$

without the Laplace transform

$$\boxed{y(t) = -\frac{1}{3}e^{-3t}}$$ General solution