Partial derivative

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If the function have more than one variable is called two or three variables in this case to drive it one can do this derivative with respect to x and y and z remain constants or with respect y then x and z remain constant and finally if the derivative with respect to z then x and y remain constant.

Example 1:

$f(x,y) = 2x^2y^3$

then , the derivative with respect to x is f_x and with respect to y is f_y so ,

$f_x = 4xy^3$ its clear that y remain constant and $f_y= 6x^2y^2$

so, the definition of both f_x and f_y are as follows:

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Then for given function f(x,y) the following are equivalent notations:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\big(f(x,y)\big) = z_x = \frac{\partial z}{\partial x} = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big(f(x,y)\big) = z_y = \frac{\partial z}{\partial y} = D_y f$$

For the fractional notation for the partial derivative notice the difference between the partial derivative and the ordinary derivative from single variable calculus.

$$f(x) \quad \Rightarrow \quad f'(x) = \frac{df}{dx}$$
$$f(x,y) \quad \Rightarrow \quad f_x(x,y) = \frac{\partial f}{\partial x} \ \ \& \ \ f_y(x,y) = \frac{\partial f}{\partial y}$$

Example 1 Find all of the first order partial derivatives for the following functions.

(a) $f(x,y) = x^4 + 6\sqrt{y} - 10$

(b) $w = x^2 y - 10y^2 z^3 + 43x - 7\tan(4y)$

(c) $h(s,t) = t^7 \ln(s^2) + \dfrac{9}{t^3} - \sqrt[7]{s^4}$

(d) $f(x,y) = \cos\left(\dfrac{4}{x}\right) \mathbf{e}^{x^2 y - 5y^3}$

Solution

(a) $f(x,y) = x^4 + 6\sqrt{y} - 10$

Let's first take the derivative with respect to *x* and remember that as we do so all the *y*'s will be treated as constants. The partial derivative with respect to *x* is,

$$f_x(x,y) = 4x^3$$

Notice that the second and the third term differentiate to zero in this case. It should be clear why the third term differentiated to zero. It's a constant and we know that constants always differentiate to

.

Now, let's take the derivative with respect to *y*. In this case we treat all *x*'s as constants and so the first term involves only *x*'s and so will differentiate to zero, just as the third term will. Here is the partial derivative with respect to *y*.

$$f_y(x,y) = \frac{3}{\sqrt{y}}$$

(b) $w = x^2 y - 10y^2 z^3 + 43x - 7\tan(4y)$

Here is the partial derivative with respect to *x*.

$$\frac{\partial w}{\partial x} = 2xy + 43$$

Let's now differentiate with respect to *y*. In this case all *x*'s and *z*'s will be treated as constants. This means the third term will differentiate to zero since it contains only *x*'s while the *x*'s in the first term and the *z*'s in the second term will be treated as multiplicative constants. Here is the derivative with respect to *y*.

$$\frac{\partial w}{\partial y} = x^2 - 20yz^3 - 28\sec^2\left(4y\right)$$

Finally, let's get the derivative with respect to *z*. Since only one of the terms involve *z*'s this will be the only non-zero term in the derivative. Also, the *y*'s in that term will be treated as multiplicative constants. Here is the derivative with respect to *z*.

$$\frac{\partial w}{\partial z} = -30y^2z^2$$

(c) $h\left(s,t\right) = t^7\ln\left(s^2\right) + \frac{9}{t^3} - \sqrt[7]{s^4}$

With this one we'll not put in the detail of the first two. Before taking the derivative let's rewrite the function a little to help us with the differentiation process.

$$h\left(s,t\right) = t^7\ln\left(s^2\right) + 9t^{-3} - s^{\frac{4}{7}}$$

$$h_s\left(s,t\right) = \frac{\partial h}{\partial s} = t^7\left(\frac{2s}{s^2}\right) - \frac{4}{7}s^{-\frac{3}{7}} = \frac{2t^7}{s} - \frac{4}{7}s^{-\frac{3}{7}}$$

$$h_t\left(s,t\right) = \frac{\partial h}{\partial t} = 7t^6\ln\left(s^2\right) - 27t^{-4}$$

Remember how to differentiate natural logarithms.

$$\frac{d}{dx}\left(\ln g\left(x\right)\right) = \frac{g'\left(x\right)}{g\left(x\right)}$$

(d) $f\left(x,y\right) = \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}$

$$\begin{aligned} f_x\left(x,y\right) &= -\sin\left(\frac{4}{x}\right)\left(-\frac{4}{x^2}\right)\mathbf{e}^{x^2y-5y^3} + \cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}\left(2xy\right) \\ &= \frac{4}{x^2}\sin\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3} + 2xy\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3} \end{aligned}$$

Also, don't forget how to differentiate exponential functions,

$$\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\mathbf{e}^{f(x)}$$

$$f_y(x,y) = \left(x^2 - 15y^2\right)\cos\left(\frac{4}{x}\right)\mathbf{e}^{x^2y-5y^3}$$

Example 2 Find all of the first order partial derivatives for the following functions.

(a) $z = \dfrac{9u}{u^2+5v}$

(b) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$

(c) $z = \sqrt{x^2 + \ln\left(5x - 3y^2\right)}$

Solution

(a) $z = \dfrac{9u}{u^2+5v}$

We also can't forget about the quotient rule. Since there isn't too much to this one, we will simply give the derivatives.

$$z_u = \frac{9\left(u^2+5v\right) - 9u(2u)}{\left(u^2+5v\right)^2} = \frac{-9u^2+45v}{\left(u^2+5v\right)^2}$$

$$z_v = \frac{(0)\left(u^2+5v\right) - 9u(5)}{\left(u^2+5v\right)^2} = \frac{-45u}{\left(u^2+5v\right)^2}$$

In the case of the derivative with respect to *v* recall that *u*'s are constant and so when we differentiate the numerator we will get zero!

(b) $g(x,y,z) = \dfrac{x\sin(y)}{z^2}$

$$g_x(x,y,z) = \frac{\sin(y)}{z^2} \qquad\qquad g_y(x,y,z) = \frac{x\cos(y)}{z^2}$$

Now, in the case of differentiation with respect to *z* we can avoid the quotient rule with a quick rewrite of the function. Here is the rewrite as well as the derivative with respect to *z*.

$$g(x,y,z) = x\sin(y)z^{-2}$$

$$g_z(x,y,z) = -2x\sin(y)z^{-3} = -\frac{2x\sin(y)}{z^3}$$

(c) $z = \sqrt{x^2 + \ln\left(5x - 3y^2\right)}$

In this last part we are just going to do a somewhat messy chain rule problem. However, if you had a good background in Calculus I chain rule this shouldn't be all that difficult of a problem. Here are the two derivatives,

$$\begin{aligned} z_x &= \frac{1}{2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \frac{\partial}{\partial x}\left(x^2 + \ln\left(5x - 3y^2\right)\right) \\ &= \frac{1}{2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \left(2x + \frac{5}{5x - 3y^2}\right) \\ &= \left(x + \frac{5}{2\left(5x - 3y^2\right)}\right)\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} z_y &= \frac{1}{2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \frac{\partial}{\partial y}\left(x^2 + \ln\left(5x - 3y^2\right)\right) \\ &= \frac{1}{2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \left(\frac{-6y}{5x - 3y^2}\right) \\ &= -\frac{3y}{5x - 3y^2}\left(x^2 + \ln\left(5x - 3y^2\right)\right)^{-\frac{1}{2}} \end{aligned}$$

Example 3 Find $\frac{dy}{dx}$ for $3y^4 + x^7 = 5x$.

Solution

Remember that the key to this is to always think of *y* as a function of *x*, or $y = y\left(x\right)$ and so whenever we differentiate a term involving *y*'s with respect to *x* we will really need to use the chain rule which will mean that we will add on a $\frac{dy}{dx}$ to that term.

The first step is to differentiate both sides with respect to *x*.

$$12y^3 \frac{dy}{dx} + 7x^6 = 5$$

The final step is to solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{5 - 7x^6}{12y^3}$$

Higher Order Partial Derivatives:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$
$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Example 1 Find all the second order derivatives for $f(x,y) = \cos(2x) - x^2 \mathbf{e}^{5y} + 3y^2$.

Solution
We'll first need the first order derivatives so here they are.

$$f_x(x,y) = -2\sin(2x) - 2x\mathbf{e}^{5y}$$
$$f_y(x,y) = -5x^2\mathbf{e}^{5y} + 6y$$

Now, let's get the second order derivatives.

$$f_{xx} = -4\cos(2x) - 2\mathbf{e}^{5y}$$
$$f_{xy} = -10x\mathbf{e}^{5y}$$
$$f_{yx} = -10x\mathbf{e}^{5y}$$
$$f_{yy} = -25x^2\mathbf{e}^{5y} + 6$$

Differentials:

Given the function $z = f(x,y)$ the differential *dz* or *df* is given by,

$$dz = f_x\,dx + f_y\,dy \qquad \text{or} \qquad df = f_x\,dx + f_y\,dy$$

There is a natural extension to functions of three or more variables. For instance, given the function $w = g(x,y,z)$ the differential is given by,

$$dw = g_x\,dx + g_y\,dy + g_z\,dz$$

Let's do a couple of quick examples.

Example 1 Compute the differentials for each of the following functions.

(a) $z = \mathbf{e}^{x^2+y^2} \tan(2x)$

(b) $u = \dfrac{t^3 r^6}{s^2}$

Solution

(a) $z = \mathbf{e}^{x^2+y^2} \tan(2x)$

There really isn't a whole lot to these outside of some quick differentiation. Here is the differential for the function.

$$dz = \left(2x\mathbf{e}^{x^2+y^2} \tan(2x) + 2\mathbf{e}^{x^2+y^2} \sec^2(2x)\right)dx + 2y\mathbf{e}^{x^2+y^2} \tan(2x)\,dy$$

(b) $u = \dfrac{t^3 r^6}{s^2}$

Here is the differential for this function.

$$du = \frac{3t^2 r^6}{s^2}dt + \frac{6t^3 r^5}{s^2}dr - \frac{2t^3 r^6}{s^3}ds$$

Note that sometimes these differentials are called the **total differentials**.

Chain Rule:

If $\quad y = f(x) \quad$ and $\quad x = g(t) \quad$ then $\quad \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$

Case 1 : $z = f(x,y)$, $x = g(t)$, $y = h(t)$ and compute $\dfrac{dz}{dt}$.

The chain rule for this case is,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 1 Compute $\frac{dz}{dt}$ for each of the following.

(a) $z = x\mathbf{e}^{xy}$, $x = t^2$, $y = t^{-1}$

(b) $z = x^2y^3 + y\cos x$, $x = \ln\left(t^2\right)$, $y = \sin\left(4t\right)$

Solution

(a) $z = x\mathbf{e}^{xy}$, $x = t^2$, $y = t^{-1}$

There really isn't all that much to do here other than using the formula.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \left(\mathbf{e}^{xy} + yx\mathbf{e}^{xy}\right)\left(2t\right) + x^2\mathbf{e}^{xy}\left(-t^{-2}\right) \\ &= 2t\left(\mathbf{e}^{xy} + yx\mathbf{e}^{xy}\right) - t^{-2}x^2\mathbf{e}^{xy}\end{aligned}$$

So, technically we've computed the derivative. However, we should probably go ahead and substitute in for *x* and *y* as well at this point since we've already got *t*'s in the derivative. Doing this gives,

$$\frac{dz}{dt} = 2t\left(\mathbf{e}^{t} + t\mathbf{e}^{t}\right) - t^{-2}t^4\mathbf{e}^{t} = 2t\mathbf{e}^{t} + t^2\mathbf{e}^{t}$$

Note that in this case it might actually have been easier to just substitute in for *x* and *y* in the original function and just compute the derivative as we normally would. For comparison's sake let's do that.

$$z = t^2\mathbf{e}^{t} \qquad \Rightarrow \qquad \frac{dz}{dt} = 2t\mathbf{e}^{t} + t^2\mathbf{e}^{t}$$

The same result for less work. Note however, that often it will actually be more work to do the substitution first.

(b) $z = x^2y^3 + y\cos x$, $x = \ln\left(t^2\right)$, $y = \sin\left(4t\right)$

Okay, in this case it would almost definitely be more work to do the substitution first so we'll use the chain rule first and then substitute.

$$\begin{aligned}\frac{dz}{dt} &= \left(2xy^3 - y\sin x\right)\left(\frac{2}{t}\right) + \left(3x^2y^2 + \cos x\right)\left(4\cos\left(4t\right)\right) \\ &= \frac{4\sin^3\left(4t\right)\ln t^2 - 2\sin\left(4t\right)\sin\left(\ln t^2\right)}{t} + 4\cos\left(4t\right)\left(3\sin^2\left(4t\right)\left[\ln t^2\right]^2 + \cos\left(\ln t^2\right)\right)\end{aligned}$$

Now, there is a special case that we should take a quick look at before moving on to the next case. Let's suppose that we have the following situation,

$$z = f(x,y) \qquad\qquad y = g(x)$$

In this case the chain rule for $\frac{dz}{dx}$ becomes,

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

In the first term we are using the fact that,

$$\frac{dx}{dx} = \frac{d}{dx}(x) = 1$$

Let's take a quick look at an example.

Example 2 Compute $\frac{dz}{dx}$ for $z = x\ln(xy) + y^3$, $y = \cos(x^2+1)$

Solution

We'll just plug into the formula.

$$\begin{aligned}
\frac{dz}{dx} &= \left(\ln(xy) + x\frac{y}{xy}\right) + \left(x\frac{x}{xy} + 3y^2\right)\left(-2x\sin(x^2+1)\right) \\
&= \ln\left(x\cos(x^2+1)\right) + 1 - 2x\sin(x^2+1)\left(\frac{x}{\cos(x^2+1)} + 3\cos^2(x^2+1)\right) \\
&= \ln\left(x\cos(x^2+1)\right) + 1 - 2x^2\tan(x^2+1) - 6x\sin(x^2+1)\cos^2(x^2+1)
\end{aligned}$$

<u>**Case 2 :**</u> $z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$ and compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

In this case if we were to substitute in for *x* and *y* we would get that *z* is a function of *s* and *t* and so it makes sense that we would be computing partial derivatives here and that there would be two of them.

Here is the chain rule for both of these cases.

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad\qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

So, not surprisingly, these are very similar to the first case that we looked at. Here is a quick example of this kind of chain rule.

Example 3 Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ for $z = \mathbf{e}^{2r}\sin(3\theta)$, $r = st - t^2$, $\theta = \sqrt{s^2+t^2}$.

Solution

Here is the chain rule for $\dfrac{\partial z}{\partial s}$.

$$\begin{aligned}\frac{\partial z}{\partial s} &= \left(2\mathbf{e}^{2r}\sin(3\theta)\right)(t) + \left(3\mathbf{e}^{2r}\cos(3\theta)\right)\frac{s}{\sqrt{s^2+t^2}} \\ &= t\left(2\mathbf{e}^{2\left(st-t^2\right)}\sin\left(3\sqrt{s^2+t^2}\right)\right) + \frac{3s\mathbf{e}^{2\left(st-t^2\right)}\cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\end{aligned}$$

Now the chain rule for $\dfrac{\partial z}{\partial t}$.

$$\begin{aligned}\frac{\partial z}{\partial t} &= \left(2\mathbf{e}^{2r}\sin(3\theta)\right)(s-2t) + \left(3\mathbf{e}^{2r}\cos(3\theta)\right)\frac{t}{\sqrt{s^2+t^2}} \\ &= (s-2t)\left(2\mathbf{e}^{2\left(st-t^2\right)}\sin\left(3\sqrt{s^2+t^2}\right)\right) + \frac{3t\mathbf{e}^{2\left(st-t^2\right)}\cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}\end{aligned}$$

Example 4 Use a tree diagram to write down the chain rule for the given derivatives.

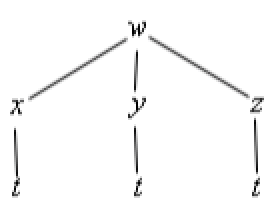
(a) $\dfrac{dw}{dt}$ for $w = f(x,y,z)$, $x = g_1(t)$, $y = g_2(t)$, and $z = g_3(t)$

(b) $\dfrac{\partial w}{\partial r}$ for $w = f(x,y,z)$, $x = g_1(s,t,r)$, $y = g_2(s,t,r)$, and $z = g_3(s,t,r)$

Solution

(a) $\dfrac{dw}{dt}$ for $w = f(x,y,z)$, $x = g_1(t)$, $y = g_2(t)$, and $z = g_3(t)$

So, we'll first need the tree diagram so let's get that.



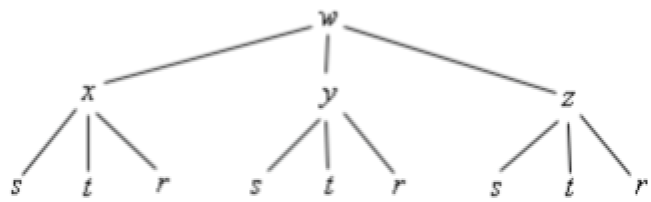
From this it looks like the chain rule for this case should be,

$$\frac{dw}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}$$

which is really just a natural extension to the two variable case that we saw above.

(b) $\dfrac{\partial w}{\partial r}$ for $w=f\left(x,y,z\right)$, $x=g_1\left(s,t,r\right)$, $y=g_2\left(s,t,r\right)$, and $z=g_3\left(s,t,r\right)$

Here is the tree diagram for this situation.



From this it looks like the derivative will be,

$$\frac{\partial w}{\partial r}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

Example 5 Compute $\dfrac{\partial^2 f}{\partial \theta^2}$ for $f\left(x,y\right)$ if $x=r\cos\theta$ and $y=r\sin\theta$.

Solution

We will need the first derivative before we can even think about finding the second derivative so let's get that. This situation falls into the second case that we looked at above so we don't need a new tree diagram. Here is the first derivative.

$$\begin{aligned}\frac{\partial f}{\partial \theta}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\\&=-r\sin\left(\theta\right)\frac{\partial f}{\partial x}+r\cos\left(\theta\right)\frac{\partial f}{\partial y}\end{aligned}$$

Okay, now we know that the second derivative is,

$$\frac{\partial^2 f}{\partial \theta^2}=\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right)=\frac{\partial}{\partial \theta}\left(-r\sin\left(\theta\right)\frac{\partial f}{\partial x}+r\cos\left(\theta\right)\frac{\partial f}{\partial y}\right)$$

The issue here is to correctly deal with this derivative. Since the two first order derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, are both functions of *x* and *y* which are in turn functions of *r* and θ both of these terms are products. So, the using the product rule gives the following,

$$\frac{\partial^2 f}{\partial \theta^2} = -r\cos(\theta)\frac{\partial f}{\partial x} - r\sin(\theta)\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) - r\sin(\theta)\frac{\partial f}{\partial y} + r\cos(\theta)\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$$

We now need to determine what $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)$ and $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$ will be. These are both chain rule problems again since both of the derivatives are functions of *x* and *y* and we want to take the derivative with respect to θ.

Before we do these let's rewrite the first chain rule that we did above a little.

$$\frac{\partial}{\partial \theta}(f) = -r\sin(\theta)\frac{\partial}{\partial x}(f) + r\cos(\theta)\frac{\partial}{\partial y}(f) \qquad (1)$$

Note that all we've done is change the notation for the derivative a little. With the first chain rule written in this way we can think of (1) as a formula for differentiating any function of *x* and *y* with respect to θ provided we have $x = r\cos\theta$ and $y = r\sin\theta$.

This however is exactly what we need to do the two new derivatives we need above. Both of the first order partial derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, are functions of *x* and *y* and $x = r\cos\theta$ and $y = r\sin\theta$ so we can use (1) to compute these derivatives.

To do this we'll simply replace all the *f*'s in (1) with the first order partial derivative that we want to differentiate. At that point all we need to do is a little notational work and we'll get the formula that we're after.

Here is the use of (1) to compute $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)$.

$$\begin{aligned}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) &= -r\sin(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + r\cos(\theta)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ &= -r\sin(\theta)\frac{\partial^2 f}{\partial x^2} + r\cos(\theta)\frac{\partial^2 f}{\partial y \partial x}\end{aligned}$$

Here is the computation for $\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$.

$$\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)=-r\sin(\theta)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)+r\cos(\theta)\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$=-r\sin(\theta)\frac{\partial^2 f}{\partial x\partial y}+r\cos(\theta)\frac{\partial^2 f}{\partial y^2}$$

The final step is to plug these back into the second derivative and do some simplifying.

$$\begin{aligned}\frac{\partial^2 f}{\partial\theta^2}&=-r\cos(\theta)\frac{\partial f}{\partial x}-r\sin(\theta)\left(-r\sin(\theta)\frac{\partial^2 f}{\partial x^2}+r\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}\right)-\\&\qquad r\sin(\theta)\frac{\partial f}{\partial y}+r\cos(\theta)\left(-r\sin(\theta)\frac{\partial^2 f}{\partial x\partial y}+r\cos(\theta)\frac{\partial^2 f}{\partial y^2}\right)\\&=-r\cos(\theta)\frac{\partial f}{\partial x}+r^2\sin^2(\theta)\frac{\partial^2 f}{\partial x^2}-r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}-\\&\qquad r\sin(\theta)\frac{\partial f}{\partial y}-r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial x\partial y}+r^2\cos^2(\theta)\frac{\partial^2 f}{\partial y^2}\\&=-r\cos(\theta)\frac{\partial f}{\partial x}-r\sin(\theta)\frac{\partial f}{\partial y}+r^2\sin^2(\theta)\frac{\partial^2 f}{\partial x^2}-\\&\qquad 2r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial y\partial x}+r^2\cos^2(\theta)\frac{\partial^2 f}{\partial y^2}\end{aligned}$$

It's long and fairly messy but there it is.

The final topic in this section is a revisiting of implicit differentiation. With these forms of the chain rule implicit differentiation actually becomes a fairly simple process. Let's start out with the implicit differentiation that we saw in a Calculus I course.

We will start with a function in the form $F(x,y)=0$ (if it's not in this form simply move everything to one side of the equal sign to get it into this form) where $y=y(x)$. In a Calculus I course we were then asked to compute $\frac{dy}{dx}$ and this was often a fairly messy process. Using the chain rule from this section however we can get a nice simple formula for doing this. We'll start by differentiating both sides with respect to *x*. This will mean using the chain rule on the left side and the right side will, of course, differentiate to zero. Here are the results of that.

$$F_x+F_y\frac{dy}{dx}=0 \qquad\Rightarrow\qquad \frac{dy}{dx}=-\frac{F_x}{F_y}$$

As shown, all we need to do next is solve for $\frac{dy}{dx}$ and we've now got a very nice formula to use for implicit differentiation. Note as well that in order to simplify the formula we switched back to using the subscript notation for the derivatives.

Let’s check out a quick example.

Example 6 Find $\frac{dy}{dx}$ for $x\cos(3y)+x^3y^5=3x-\mathbf{e}^{xy}$.

Solution
The first step is to get a zero on one side of the equal sign and that’s easy enough to do.

$$x\cos(3y)+x^3y^5-3x+\mathbf{e}^{xy}=0$$

Now, the function on the left is $F(x,y)$ in our formula so all we need to do is use the formula to find the derivative.

$$\frac{dy}{dx}=-\frac{\cos(3y)+3x^2y^5-3+y\mathbf{e}^{xy}}{-3x\sin(3y)+5x^3y^4+x\mathbf{e}^{xy}}$$

There we go. It would have taken much longer to do this using the old Calculus I way of doing this.

We can also do something similar to handle the types of implicit differentiation problems involving partial derivatives like those we saw when we first introduced partial derivatives. In these cases we will start off with a function in the form $F(x,y,z)=0$ and assume that $z=f(x,y)$ and we want to find $\frac{\partial z}{\partial x}$ and/or $\frac{\partial z}{\partial y}$.

Let’s start by trying to find $\frac{\partial z}{\partial x}$. We will differentiate both sides with respect to *x* and we’ll need to remember that we’re going to be treating *y* as a constant. Also, the left side will require the chain rule. Here is this derivative.

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

Now, we have the following,

$$\frac{\partial x}{\partial x}=1 \qquad \text{and} \qquad \frac{\partial y}{\partial x}=0$$

The first is because we are just differentiating *x* with respect to *x* and we know that is 1. The second is because we are treating the *y* as a constant and so it will differentiate to zero.

Plugging these in and solving for $\frac{\partial z}{\partial x}$ gives,

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

A similar argument can be used to show that,

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

As with the one variable case we switched to the subscripting notation for derivatives to simplify the formulas. Let's take a quick look at an example of this.

Example 7 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2 \sin(2y-5z) = 1 + y\cos(6zx)$.

Solution

This was one of the functions that we used the old implicit differentiation on back in the Partial Derivatives section. You might want to go back and see the difference between the two.

First let's get everything on one side.

$$x^2 \sin(2y-5z) - 1 - y\cos(6zx) = 0$$

Now, the function on the left is $F(x,y,z)$ and so all that we need to do is use the formulas developed above to find the derivatives.

$$\frac{\partial z}{\partial x} = -\frac{2x\sin(2y-5z) + 6yz\sin(6zx)}{-5x^2\cos(2y-5z) + 6yx\sin(6zx)}$$

$$\frac{\partial z}{\partial y} = -\frac{2x^2\cos(2y-5z) - \cos(6zx)}{-5x^2\cos(2y-5z) + 6yx\sin(6zx)}$$

If you go back and compare these answers to those that we found the first time around you will notice that they might appear to be different. However, if you take into account the minus sign that sits in the front of our answers here you will see that they are in fact the same.

Directional Derivatives:

The rate of change of $f(x,y)$ in the direction of the unit vector $\vec{u} = \langle a, b\rangle$ is called the **directional derivative** and is denoted by $D_{\vec{u}}f(x,y)$. The definition of the directional derivative is,

$$D_{\vec{u}}f(x,y) = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$

$D_{\vec{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$ ……………………………………………..2

$D_{\vec{u}}f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c$ …………………….3

Example 1 Find each of the directional derivatives.

(a) $D_{\vec{u}} f(2,0)$ where $f(x,y) = x\mathbf{e}^{xy} + y$ and $\vec{u}$ is the unit vector in the direction of $\theta = \frac{2\pi}{3}$.

(b) $D_{\vec{u}} f(x,y,z)$ where $f(x,y,z) = x^2 z + y^3 z^2 - xyz$ in the direction of $\vec{v} = \langle -1, 0, 3 \rangle$.

Solution

(a) $D_{\vec{u}} f(2,0)$ **where** $f(x,y) = x\mathbf{e}^{xy} + y$ **and** $\vec{u}$ **is the unit vector in the direction of** $\theta = \frac{2\pi}{3}$**.**

We'll first find $D_{\vec{u}} f(x,y)$ and then use this a formula for finding $D_{\vec{u}} f(2,0)$. The unit vector giving the direction is,

$$\vec{u} = \left\langle \cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right) \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

So, the directional derivative is,

$$D_{\vec{u}} f(x,y) = \left(-\frac{1}{2}\right)\left(\mathbf{e}^{xy} + xy\mathbf{e}^{xy}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(x^2\mathbf{e}^{xy} + 1\right)$$

Now, plugging in the point in question gives,

$$D_{\vec{u}} f(2,0) = \left(-\frac{1}{2}\right)(1) + \left(\frac{\sqrt{3}}{2}\right)(5) = \frac{5\sqrt{3}-1}{2}$$

(b) $D_{\vec{u}} f(x,y,z)$ **where** $f(x,y,z) = x^2 z + y^3 z^2 - xyz$ **in the direction of** $\vec{v} = \langle -1, 0, 3 \rangle$**.**

In this case let's first check to see if the direction vector is a unit vector or not and if it isn't convert it into one. To do this all we need to do is compute its magnitude.

$$\|\vec{v}\| = \sqrt{1+0+9} = \sqrt{10} \neq 1$$

So, it's not a unit vector. Recall that we can convert any vector into a unit vector that points in the same direction by dividing the vector by its magnitude. So, the unit vector that we need is,

$$\vec{u} = \frac{1}{\sqrt{10}}\langle -1, 0, 3 \rangle = \left\langle -\frac{1}{\sqrt{10}}, 0, \frac{3}{\sqrt{10}} \right\rangle$$

The directional derivative is then,

$$\begin{aligned} D_{\vec{u}} f(x,y,z) &= \left(-\frac{1}{\sqrt{10}}\right)(2xz - yz) + (0)\left(3y^2z^2 - xz\right) + \left(\frac{3}{\sqrt{10}}\right)\left(x^2 + 2y^3 z - xy\right) \\ &= \frac{1}{\sqrt{10}}\left(3x^2 + 6y^3 z - 3xy - 2xz + yz\right) \end{aligned}$$