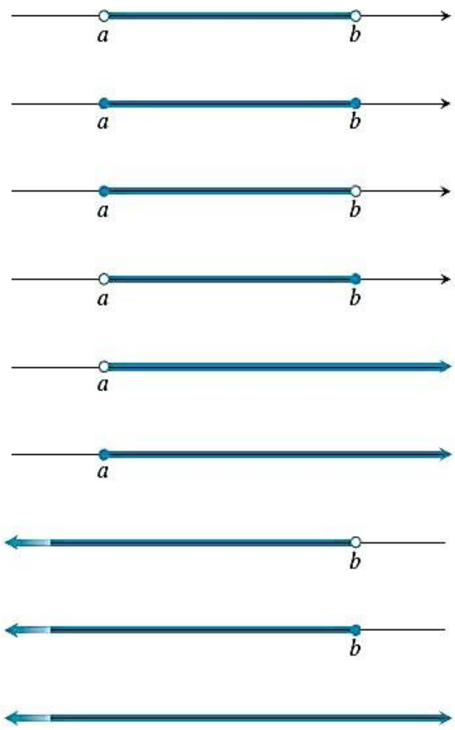
Real Numbers

Intervals:

The interval is a subset of real numbers $\Re$ and contains three types

- open interval (a,b)
- closed interval [a,b]
- half closed and half open (a,b] , [a,b)

$(a, b) = \{x \mid a < x < b\}$	Open	
$[a, b] = \{x \mid a \leq x \leq b\}$	Closed	
$[a, b) = \{x \mid a \leq x < b\}$	Half-open	
$(a, b] = \{x \mid a < x \leq b\}$	Half-open	
$(a, \infty) = \{x \mid x > a\}$	Open	
$[a, \infty) = \{x \mid x \geq a\}$	Closed	
$(-\infty, b) = \{x \mid x < b\}$	Open	
$(-\infty, b] = \{x \mid x \leq b\}$	Closed	
$(-\infty, \infty) = \mathcal{R}$ (set of all real numbers)	Both open	



Real functions

Function: a relationship between the two groups so that each element of the domain group is linked to only a single element of a group

The opposite field and write $y = f(x)$

Domain group values (the possible values for x) Domain for x .

Range : which is a subset of the domain group contrast (the possible values for y with respect to values for x)

Example:

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution:The formula $y = x^2$ gives a real *y*-value for any real number *x*, so the domainis$(-\infty, \infty)$. The range of $y = x^2$ is$[0, \infty)$ because the square of any real number isnonnegative and every nonnegative number *y* is the square of its own square root, $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real *y*-value for every *x* except *x*= 0*We cannot divide any number by zero*. The range of $y = 1/x$ the set of reciprocals of all nonzero real numbers,is the set of all nonzero real numbers, since $y = 1/(1/y)$.

The formula $y = \sqrt{x}$ gives a real *y*-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is$[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$ the quantity 4 -*x*cannot be negative. That is, 4 -*x* ≥ 0 ,or*x* ≤ 4

The formula gives real *y*-values for all*x*≤ 4The range of is$[0, \infty)$,

the set of all nonnegative numbers.

The finalformula gives a real *y*-value for every *x* in the closed interval

From -1 to 1. Outside this domain,$1 - x^2$is negative and its square root is not a real number. The values of$1 - x^2$vary from 0 to 1 on the given domain, and the square rootsof these values do the same. The range of $\sqrt{1 - x^2}$ is [0, 1].

Limits

We say the real number L is the limit of the function f when $x \to x_0$ as written $\lim\limits_{x \to x_0} f(x) = L$.

Note that the limit Spread over inside the parentheses of summation or subtract, multiply, divide for two functions .

Example: Evaluate the limit

$$1.\ \lim_{x \to 1} \frac{2x^4 - 6x^3 + x^2 + 3}{x - 1}$$

$$= \lim_{x \to 1} 2x^3 - 4x^2 - 3x - 3 = -8$$

$$\begin{array}{r|l} & 2x^3 - 4x^2 - 3x - 3 \\ \hline x - 1 & 2x^4 - 6x^3 + x^2 + 3 \\ & \underline{\mp 2x^4 \pm 2x^3} \\ & -4x^3 + x^2 + 3 \\ & \underline{\pm 4x^3 \mp 4x^2} \\ & -3x^2 + 3 \\ & \underline{\pm 3x^2 \mp 3x} \\ & -3x + 3 \\ & \underline{\pm 3x \mp 3} \\ & 0 \end{array}$$

$$2.\ \lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x} \times \frac{\sqrt{4 + x} + 2}{\sqrt{4 + x} + 2} = \lim_{x \to 0} \frac{4 + x - 4}{x\sqrt{4 + x} + 2x} = \lim_{x \to 0} \frac{x}{x\sqrt{4 + x} + 2} = \frac{1}{4}$$

Exercises: Evaluate the limits H.W

- $\lim\limits_{x \to 2} \dfrac{x^4 - 16}{x - 2}$
- $\lim\limits_{x \to -1} \dfrac{x^3 - x^2 - 5x - 2}{(x + 1)^2}$
- $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{1 + x} - 1}{x}$

Continuity

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three
conditions.

1. $f(c)$ exists (c lies in the domain of f)

2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)

3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Example: A Continuous Extension

Show that $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$

has a continuous extension to $x = 2$ and find that extension.

Solution: Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$

The new function $F(x) = \frac{x+3}{x+2}$

is equal to $f(x)$ for $x \neq 2$ but is continuous at $x = 2$ having there the value of 5/4. Thus F is the continuous extension of f to $x = 2$ and

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

Derivative

The derivative of function $f(x)$ with respect to x is $f'(x)$ as define

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad .$$

Rules of derivatives :

Let $f(x)$ and $g(x)$ are differentiable functions and let c is a constant , $n \in \Re$,then

- $\dfrac{d}{dx} c = 0$
- $\dfrac{d}{dx}[\, f(x) \mp g(x)] = f'(x) \mp g'(x)$
- $\dfrac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
- $\dfrac{d}{dx}[\dfrac{f(x)}{g(x)}] = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$; $g(x) \neq 0$
- $\dfrac{d}{dx}[f(x)]^n = n\,[f(x)]^{n-1} \cdot f'(x)$

Example: find the derivative for the function

$\cdot\ f(x) = (\dfrac{1+3x}{3x})(3-x)$

Sol: $f'(x) = (\dfrac{1+3x}{3x}) \times (-1) + (3-x) \times \dfrac{3x \times 3 - (1+3x) \times 3}{9x^2}$

$$= \frac{-1-3x}{3x} + \frac{x-3}{3x^2} = \frac{-x-3x^2+x-3}{3x^2} = \frac{-(x^2+1)}{x^2}$$

Chain rule

If $y = f(u)$ and $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

If $y = f(t)$ and $x = g(t)$ then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

Example :

Find $\dfrac{dy}{dt}$

$x = 2t - 5$ and $y = x^2 + \frac{x}{2}$

Sol:

$\frac{dy}{dx} = 2x + \frac{1}{2}$, $\frac{dx}{dt} = 2$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \rightarrow \frac{dy}{dt} = (2x + \frac{1}{2}) \times 2 = 4x + 1 = 8t - 19$.

Example :

Find $\frac{dy}{dx}$ if $x = t + \frac{1}{t}$ $y = t - \frac{1}{t}$

Sol:

$\frac{dy}{dt} = 1 + \frac{1}{t^2}$, $\frac{dx}{dt} = 1 - \frac{1}{t^2}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \rightarrow \frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{\frac{t^2+1}{t^2}}{\frac{t^2-1}{t^2}} = \frac{t^2+1}{t^2-1}$$

Exercises: H.W

Find dy/dx and by using chain rule

1. $y = x^2\sqrt{2x^2 + 3}$

2. $y = 4x\sqrt{x + \sqrt{x}}$

3. $y = \left(\frac{\sqrt{x}}{1 + x}\right)^2$

4. $y = t - \frac{1}{t^2}$, $x = t + \frac{1}{t^2}$

5. $y = \sqrt{t^2 - 4} + 3$, $x = 3t\sqrt{2t + 1}$