Taylor and Maclaurin Series

DEFINITIONS **Taylor Series, Maclaurin Series**

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

Example : Find Taylor series of $f(x) = e^x$ **at** $x = 1$ **.**

Sol:

Note that $f^{(n)}(x) = e^x$ **for all** n **, thus** $f^{(n)}(1) = e$ **for all** n

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} e$$

$$= e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \ldots\right)$$

Example : Finding a Taylor Series

$f(x) = 1/x$ **at** $a = 2$

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$f(x) = x^{-1}, \qquad f(2) = 2^{-1} = \frac{1}{2},$$

$$f'(x) = -x^{-2}, \qquad f'(2) = -\frac{1}{2^2},$$

$$f''(x) = 2!x^{-3}, \qquad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3},$$

$$f'''(x) = -3!x^{-4}, \qquad \frac{f'''(2)}{3!} = -\frac{1}{2^4},$$

$$\vdots \qquad \qquad \vdots$$

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)}, \qquad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

Example :

Find Maclaurin series for $f(x) = \sin x$

$f(x) = \sin x \rightarrow f(0) = 0$ and $f'(x) = \cos x \rightarrow f'(0) = 1$

$f'(x) = -\sin x \rightarrow f'(0) = 0$ and $f''(x) = -\cos x \rightarrow f''(0) = -1$

$f^{(4)}(x) = \sin x \rightarrow f^{(4)}(0) = 0$ and $f^{(5)}(x) = \cos x \rightarrow f^{(5)}(0) = 1$

Maclaurin series for $f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Exercises: Find the Taylor series for the function at point a

- $f(x) = \cos x$, $a = -\frac{\pi}{4}$
- $f(x) = \sqrt{x}$ $a = 4$

H.W

- $f(x) = \frac{1}{x}$, $a = -1$
- $f(x) = \ln x$, $a = 1$