Fourier Series

A function $f(x)$ is said to have period P if $f(x+P)=f(x)$ for all (x).

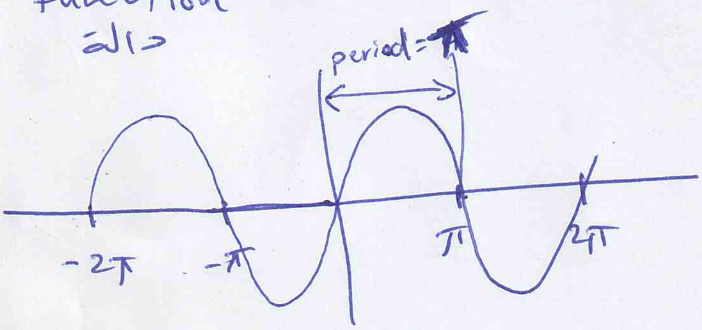
Let $f(x)$ has period 2π, it is enough to consider behavior of the function on the interval $[-\pi,\pi]$.

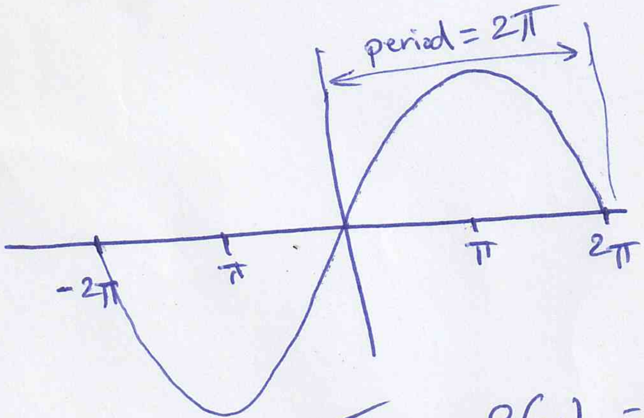
قيمة (P) تسمى دورة الدالة $f(x)$ عندما تضاف الى (X)

Ex/ $f(x)=\sin(x)$

periodic function with $P=2\pi$, where $\sin(x+2\pi)=\sin(x)$

دورية دالة





يقصد بتحليل فوريه للدالة $f(x)$ هو كتابة هذه الدالة في صورة متسلسلة من sin , cos للمتغير المستقل (X)

Take $f_1(x) = \cos x$

$f_2(x) = \sin x$

The above functions form a complete orthogonal system over $[-\pi, \pi]$, $\therefore$ the Fourier series will be:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

في بعض الأحيان نستخدم صيغ مختلفة عن a_0, a_n, b_n

$$d_n = \sqrt{a_n^2 + b_n^2}$$

$$\tan \varphi_n = \frac{a_n}{b_n}$$

$$\tan \theta_n = \frac{b_n}{a_n}$$

عندما نستخدم d_n , φ_n , θ_n بدلاً عن a_0 , a_n , b_n تكون متسلسلة فورييه كما يلي

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} d_n \sin(nx + \varphi_n)$$

or

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} d_n \cos(nx + \theta_n)$$

Ex1/

Find the Fourier Series for the square 2π-periodic wave defined on the interval $[-\pi, \pi]$

$$f(x) = \begin{cases} 0, & \text{if } -\pi \leq x \leq 0 \\ 1, & \text{if } 0 < x \leq \pi \end{cases}$$

Sol/ إيجاد الثابت a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = \frac{1}{\cancel{\pi}}[\cancel{\pi}] = \boxed{1}$$

نجد معاملات فورييه لـ $n \neq 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

مهما كانت قيمة (n) فهي تساوي صفر

$$= \frac{1}{\pi}\int_{0}^{\pi} 1 \cos(nx)\,dx = \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi}$$

$$= \frac{1}{\pi n}\cdot 0 = \boxed{0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} 1 \sin(nx)\,dx = \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi}$$

$$= -\frac{1}{\pi n}\left(\cos(n\pi) - \cos(0)\right)$$

$$= \boxed{\frac{1 - \cos(n\pi)}{\pi n}}$$

since $\because \cos(n\pi) = (-1)^n$

$$\therefore \quad b_n = \boxed{\frac{1-(-1)^n}{\pi n}}$$

ولهذا , متسلسلة فوريير لهذه الموجة هي

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi n} \sin(nx)}$$

من المعادلة العامة اعلاه نستطيع ايجاد أول ٥ حدود

$$f(x) = \frac{1}{2} + \frac{1-(-1)}{\pi} \sin x + \frac{1-(-1)^2}{2\pi} \sin 2x + \frac{1-(-1)^3}{3\pi} \sin 3x$$

$$+ \frac{1-(-1)^4}{4\pi} \sin 4x + \frac{1-(-1)^5}{5\pi} \sin 5\pi + \dots$$

Ex 2/ Find the Fourier Series for the function $f(x)$ in the period $(-\pi, \pi)$

$$f(x) = \begin{cases} 0 & ; \ -\pi < x < 0 \\ x & ; \ 0 < x < \pi \end{cases}$$

Then prove that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + L = \frac{\pi^2}{8}$$

Sol/

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} x\, dx \right] = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \boxed{\frac{\pi}{2}}$$

$$\boxed{a_n = \begin{cases} \dfrac{-2}{\pi n^2} & ; \text{ for odd } n \quad \text{فردي} \\ 0 & , \text{ for even } n \quad \text{زوجي} \end{cases}}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin(nx)\, dx = \frac{1}{\pi} \left\{ \left[\left(\frac{-x \cos(nx)}{n} \right)_0^{\pi} \right] + \frac{1}{n} \int_0^{\pi} \cos(nx)\, dx \right\}$$

$$= \frac{-\cos n\pi}{n} = \frac{-(-1)^n}{n} = \boxed{\frac{(-1)^{n+1}}{n}}$$

ستكون مسلسلة فوريه هي

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{-2}{\pi n^2} \cos(nx) \right) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n}$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + L\right)$$

$$+ \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - L\right)$$

$\boxed{x=0}$ نفرض

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + L\right)$$

$$\therefore \boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + L = \frac{\pi^2}{8}}$$

Ans.