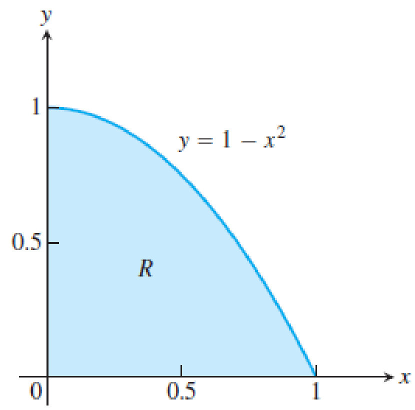
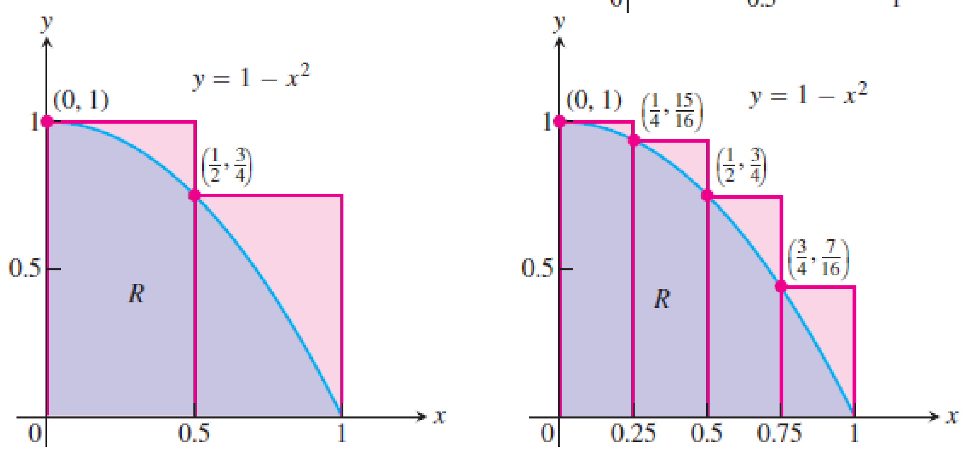
<u>*Integration*</u>

Approximating Area

What is the area of the shaded region R that lies above the x-axis, below the $y = 1 - x^2$, and between the vertical lines x = 0 and x = 1?



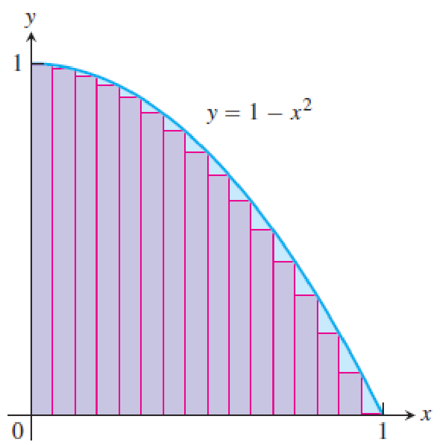
The method:-



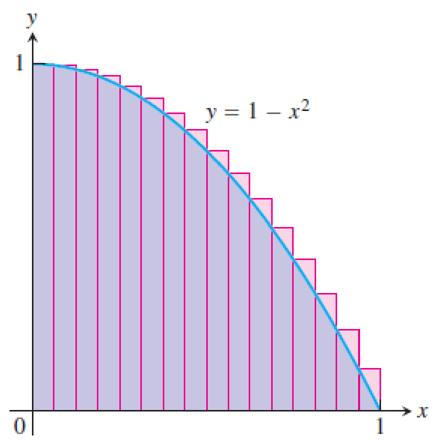
(a) (b)

$$A_a \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875$$

$$A_b \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125$$



or



$$A = \sum_0^1 A_n dx \quad \Rightarrow \quad A = \int_0^1 f(x)dx$$

The Indefinite Integration:

Some properties of integration:

1. *Constant Multiple:*

$$\int kf(x)\,dx = k\int f(x)\,dx \qquad \text{Any Number } k$$

$$\int -f(x)\,dx = -\int f(x)\,dx \qquad k = -1$$

2. *Sum and Difference:*

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Rules of Integration

1. $\int dx = x + C \quad , \int a\,dx = a\int dx = ax + C$

Example : Find the following integral.

(i) $\int 6dx = 6\int dx = 6x + C$

(ii) $\int \sqrt[3]{5}\,dx = \sqrt[3]{5}\int dx = \sqrt[3]{5}\,x + C$

(iii) $\int ln(8)\,dx = ln(8)\int dx = ln(8)\,x + C$

2. $\int a\,x^n dx = a\int x^n dx = \dfrac{a\,x^{n+1}}{n+1} + C \quad , n \neq -1$

Example : Find the following integral.

(i) $\int 3x^4\,dx = \dfrac{3x^{4+1}}{4+1} + c = \dfrac{3}{5}x^5 + c$

(ii) $\int \dfrac{2}{x^2}\,dx = \int 2x^{-2}\,dx = \dfrac{2x^{-2+1}}{-2+1} + c = \dfrac{2x^{-1}}{-1} + c = \dfrac{-2}{x} + c$

(iii) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \dfrac{2}{3}\sqrt{x^3} + c$

Example : $\int (3x + 2x^2 - 5)\,dx$

$$= \int 3x\,dx + \int 2x^2\,dx - \int 5\,dx \;= \mathbf{\frac{3x^2}{2} + \frac{2x^3}{3} - 5x + c}$$

Example : $\int \frac{2x^3 - 3x}{4x}\,dx$

$$= \int \left(\frac{2x^3}{4x} - \frac{3x}{4x}\right)dx = \int \left(\frac{x^2}{2} - \frac{3}{4}\right)dx$$

$$= \left(\frac{1}{2}\right)\frac{x^{2+1}}{2+1} - \frac{3}{4}x + c \;= \left(\frac{1}{2}\right)\frac{x^3}{3} - \frac{3}{4}x + c = \mathbf{\frac{1}{6}x^3 - \frac{3}{4}x + c}$$

Example: $\int (1 - t)^2\,dt$

$$\int (1 - 2t + t^2)\,dt = t - \frac{2t^{1+1}}{1+1} + \frac{t^{2+1}}{2+1} + c$$

$$= t - \frac{2t^2}{2} + \frac{t^3}{3} + c \;= \mathbf{t - t^2 + \frac{1}{3}t^3 + c}$$

H.W.: Calculate

1. (a) $\int 4\,dx$ (b) $\int 7x\,dx$ $\left[(a)\ 4x + c \quad (b)\ \frac{7x^2}{2} + c\right]$	2. (a) $\int \frac{2}{5}x^2\,dx$ (b) $\int \frac{5}{6}x^3\,dx$ $\left[(a)\ \frac{2}{15}x^3 + c \quad (b) \frac{5}{24}x^4 + c\right]$
3. (a) $\int\left(\frac{3x^2 - 5x}{x}\right)dx$ (b) $\int (2+\theta)^2\,d\theta$ $\left[(a)\ \frac{3x^2}{2} - 5x + c \quad (b)\ 4\theta + 2\theta^2 + \frac{\theta^3}{3} + c\right]$	4. (a) $\int \frac{4}{3x^2}\,dx$ (b) $\int \frac{3}{4x^4}\,dx$ $\left[(a)\ \frac{-4}{3x} + c\ (b)\ \frac{-1}{4x^3} + c\right]$
5. (a) $2\int \sqrt{x^3}\,dx$ (b) $\int \frac{1}{4}\sqrt[4]{x^5}\,dx$ $\left[(a)\ \frac{4}{5}\sqrt{x^5} + c \quad (b)\ \frac{1}{9}\sqrt[4]{x^9} + c\right]$	6. (a) $\int \frac{-5}{\sqrt{t^3}}\,dt$ (b) $\int \frac{3}{7\sqrt[5]{x^4}}\,dx$ $\left[(a)\ \frac{10}{\sqrt{t}} + c \quad (b)\ \frac{15}{7}\sqrt[5]{x} + c\right]$
7. $\int \frac{(1+\theta)^2}{\sqrt{\theta}}\,d\theta.$	8. $\int\left(4 + \frac{3}{7}x - 6x^2\right)dx.$

3. $\int (f(x))^n f'(x)dx = \frac{(f(x))^{n+1}}{n+1} + c \quad , n \neq -1$

Example : $\int (x^2-1)^3 2x\,dx$ ⇨ $[\, f(x)= x^2-1 \;, f'(x)= 2x]$

$= \frac{(x^2-1)^{3+1}}{3+1} + C = \frac{(x^2-1)^4}{4} + C$

Example : $\int \sqrt{1+x^2}\; 2x\,dx$ ⇨ $[\, f(x)=1+x^2 \;,\; f'(x)= 2x]$

$= \frac{(1+x^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{(1+x^2)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\,(1+x^2)^{\frac{3}{2}} + C = \frac{2\sqrt{(1+x^2)^3}}{3} + C$

Example: $\int \frac{2x^2+1}{(2x^3+3x-4\,)^5}\,dx$

$= \int (2x^3+3x-4\,)^{-5}\,(2x^2+1)dx * \frac{3}{3}$

$= \frac{1}{3}\int (2x^3+3x-4\,)^{-5}\,(6x^2+3)dx = \frac{1}{3}\frac{(2x^3+3x-4\,)^{-5+1}}{-5+1} + C$

$= \frac{(2x^3+3x-4\,)^{-4}}{-12} + C = = \frac{-1}{12(2x^3+3x-4\,)^4} + C$

4. $\int \frac{1}{x}dx = ln(x) + C \quad , \int \frac{1}{f(x)} f'(x)dx = ln(f(x)) + C$

Example : $\int \frac{4}{3x}\,dx = \frac{4}{3}\int \frac{1}{x}\,dx = \frac{4}{3} ln(x) + C$

Example: $\int \frac{2x^2+1}{2x^3+3x-4}\,dx$

$= \int \frac{2x^2+1}{2x^3+3x-4}\,dx * \frac{3}{3} = \frac{1}{3}\int \frac{6x^2+3}{2x^3+3x-4}\,dx$

$= \frac{1}{3} ln(2x^3+3x-4) + C$

5. $\int e^x dx = e^x + C$, $\int a^x dx = a^x/\ln a + C$

Example : $\int e^{2x^3-1} \; x^2 dx = \frac{1}{6}\int e^{2x^3-1} \; 6x^2 dx$

$= \frac{1}{6} e^{2x^3-1} + C$

Example : $\int 3^{\tan\frac{x}{4}} \sec^2\left(\frac{x}{4}\right) dx = 4\int 3^{\tan\frac{x}{4}} \sec^2\left(\frac{x}{4}\right)\frac{1}{4} dx$

$= 4 * 3^{\tan\frac{x}{4}} / \ln 3 + C$

<u>H.W. :</u> *Calculate*

1. (a) $\int \frac{3}{4} e^{2x} dx$ (b) $\frac{2}{3}\int \frac{dx}{e^{5x}}$ $\left[\text{(a) } \frac{3}{8} e^{2x} + c \quad \text{(b) } \frac{-2}{15 e^{5x}} + c\right]$	2. (a) $\int \frac{2}{3x} dx$ (b) $\int\left(\frac{u^2-1}{u}\right) du$ $\left[\text{(a) } \frac{2}{3}\ln x + c \quad \text{(b) } \frac{u^2}{2} - \ln u + c\right]$
3. (a) $\int \frac{(2+3x)^2}{\sqrt{x}} dx$ (b) $\int\left(\frac{1}{t} + 2t\right)^2 dt$ $\left[\text{(a) } 8\sqrt{x} + 8\sqrt{x^3} + \frac{18}{5}\sqrt{x^5} + c \quad \text{(b) } -\frac{1}{t} + 4t + \frac{4t^3}{3} + c\right]$	**4.** $\int (2x-5)^7 dx$ **5.** $\int \frac{x}{2+3x^2} dx$
6. $\int 4^{2x^3-2x}(3x^2-1)dx$	**7.** $\int\left(\frac{u}{u^2-1}\right) du$
8. $\int 5^{\sin 2x} \cos 2x \, dx$	**9.** $\int \frac{\sec^2(2x)}{(\tan 2x - 4)^3} dx$
10. $\int \frac{x^2-1}{x^3-3x} dx$	**11.** $\int 2x(2x^2-3)^5$

6. Integration of trigonometric functions

$\int \sin x\, dx = -\cos x + C$	$\int \cos x\, dx = \sin x + C$
$\int \sec^2 x\, dx = \tan x + C$	$\int \csc^2 x\, dx = -\cot x + C$
$\int \sec x \tan x\, dx = \sec x + C$	$\int \csc x \cot x\, dx = -\csc x$

Example :

1. $\int \sin(3x^2)\, x\, dx = \int \sin(3x^2)\, x\, dx * \frac{6}{6} = \frac{1}{6}\int \sin(3x^2)\, 6x\, dx$

$= \frac{-1}{6} \cos(3x^2) + C$

2. $\int \frac{\csc^2(\sqrt{x})}{\sqrt{x}} dx = \int \frac{\csc^2(\sqrt{x})}{\sqrt{x}} dx * \frac{2}{2} = 2\int \frac{\csc^2(\sqrt{x})}{2\sqrt{x}} dx$

$= -2 \cot(\sqrt{x}) + C$

3. $\int \cos^4(x) \sin(x)\, dx = \int \cos^4(x) \sin(x)\, dx\ \frac{-1}{-1}$

$= -\int \cos^4(x) * -\sin(x)\, dx = -\frac{\cos^{4+1}(x)}{4+1} + C = -\frac{\cos^5(x)}{5} + C$

4. $\int \sec^4(2x) \tan(2x)\, dx = \int \sec^3(2x)\ \sec(2x) \tan(2x)\, dx * \frac{2}{2}$

$= \frac{1}{2}\int \sec^3(2x)\ \sec(2x)\ \tan(2x)\ 2dx = \frac{\sec^4(2x)}{4} + C$

5. $\int \sin^2(x) dx = \int \frac{1}{2}(1 - \cos(2x)) dx = \frac{1}{2}\{ \int 1\, dx -$

$\int \cos(2x)\, dx \} = \frac{1}{2}\{ x - \frac{1}{2}\sin(2x) \} + C = \frac{1}{2} x - \frac{1}{4}\sin(2x) + C$

Example : Determine.

1) $\int \cos(3\theta - 1)d\theta$

2) $\int x \cdot \sin(2x^2)\,dx$

3) $\int \cos^2(2y) \cdot \sin(2y)\,dy$

4) $\int \sec^3 x \cdot \tan x \; dx$

5) $\int \sqrt{2 + \sin 3t} \cdot \cos 3t \; dt$

6) $\int \frac{d\theta}{\cos^2\theta}$

7) $\int (1 - \sin^2 3t) \cdot \cos 3t \; dt$

8) $\int \tan^3(5x) \cdot \sec^2(5x) \; dx$

9) $\int \sin^4 x \cdot \cos^3 x \; dx$

10) $\int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx$

Solution :

1) $\frac{1}{3}\int 3\cos(3\theta - 1)d\theta = \frac{1}{3}\sin(3\theta - 1) + c$

2) $\frac{1}{4}\int 4x \cdot \sin(2x^2)dx = -\frac{1}{4}\cos(2x^2) + c$

3) $-\frac{1}{2}\int (\cos 2y)^2 \cdot (-2\sin 2y \; dy) = -\frac{1}{2} \cdot \frac{(\cos 2y)^3}{3} + c = -\frac{1}{6}(\cos 2y)^3 + c$

4) $\int \sec^2 x \cdot (\sec x \cdot \tan x \cdot dx) = \frac{\sec^3 x}{3} + c$

5) $\frac{1}{3}\int (2 + \sin 3t)^{1/2}(3\cos 3t \; dt) = \frac{1}{3} \cdot \frac{(2 + \sin 3t)^{3/2}}{3/2} + c = \frac{2}{9}\sqrt{(2 + \sin 3t)^3} + c$

6) $\int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta \cdot d\theta = \tan\theta + c$

7) $\int (1 - \sin^2 3t) \cdot \cos 3t \; dt = \frac{1}{3}\int 3\cos 3t \; dt - \frac{1}{3}\int (\sin 3t)^2 \cdot 3\cos 3t$

$= \frac{1}{3}\sin 3t - \frac{1}{3} \cdot \frac{\sin^3 3t}{3} + c = \frac{1}{3} \cdot \sin 3t - \frac{1}{9}\sin^3 3t + c$

8) $\frac{1}{5}\int \tan^3 5x \cdot (5\sec^2 5x \; dx) = \frac{1}{5} \cdot \frac{\tan^4 5x}{4} + c = \frac{1}{20}\tan^4 5x + c$

9) $\int \sin^4 x \cdot \cos^3 x \; dx = \int \sin^4 x \cdot (1 - \sin^2 x) \cdot \cos x \; dx$

$= \int \sin^4 x \cdot \cos x \; dx - \int \sin^6 x \cdot \cos x \; dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$

$$10)\ \int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\csc^2\sqrt{x}-1}{\sqrt{x}}\,dx = 2\int \frac{\csc^2\sqrt{x}}{2\sqrt{x}} - \int x^{-1/2}\,dx$$

$$= 2\left(-\cot\sqrt{x}\right) - \frac{x^{1/2}}{1/2} + c = -2\cot\sqrt{x} - 2\sqrt{x} + c$$

***H.W.*:**

$\int \cos^2(x)dx$	$\int 4\cos 3x\,dx$	$\int 7\sec^2 4t\,dt$
$\int \frac{4}{3}\sec 4t\tan 4t\,dt$	$\int e^x \cdot \sin e^x\,dx$	$\int \frac{\sin(\ln x)}{x}$ dx
$\int \cot(2x+1)\cdot\csc^2(2x+1)\,dx$		

7 . Integration of inverse trigonometric functions

$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c \quad ; \quad \forall u^2 < a^2$
$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$
$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left\lvert\frac{u}{a}\right\rvert + c = -\frac{1}{a}\csc^{-1}\left\lvert\frac{u}{a}\right\rvert + c \quad ; \quad \forall u^2 > a^2$

Example :

$$1)\ \int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3} + c$$

$$2)\ \int \frac{x^2}{\sqrt{1-x^6}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1-(x^3)^2}}\left(3x^2\,dx\right) = \frac{1}{3}\sin^{-1}x^3 + c$$

$$3)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx = \sin^{-1}(\tan x) + c$$

4) $\int \frac{1}{5+x^2}\,dx = \int \frac{1}{(\sqrt{5})^2+x^2}\,dx$

$= \frac{1}{\sqrt{5}} \tan^{-1}(\frac{x}{\sqrt{5}}) + c$

5) $\int \frac{1}{\sqrt{x^2(x^2-\frac{36}{25})}}\,dx = \int \frac{1}{|x|\sqrt{x^2-(\frac{6}{5})^2}}\,dx = \frac{1}{6/5} \sec^{-1}(\frac{x}{6/5}) + c$

$= \frac{5}{6} \sec^{-1}(\frac{5x}{6}) + c$

H.W.: Finding the following integration.

1. $\int \frac{1}{1+\theta^2}\,d\theta$

2. $\int \frac{dx}{\sqrt{16-x^2}}$

3. $\int \frac{1}{49+x^2}\,dx$

4. $\int \frac{dt}{0.25+t^2}$

5. $\int \frac{du}{\sqrt{u^2(u^2-1)}}$

6. $\int \frac{1}{|x|\sqrt{x^2-41}}\,dx$

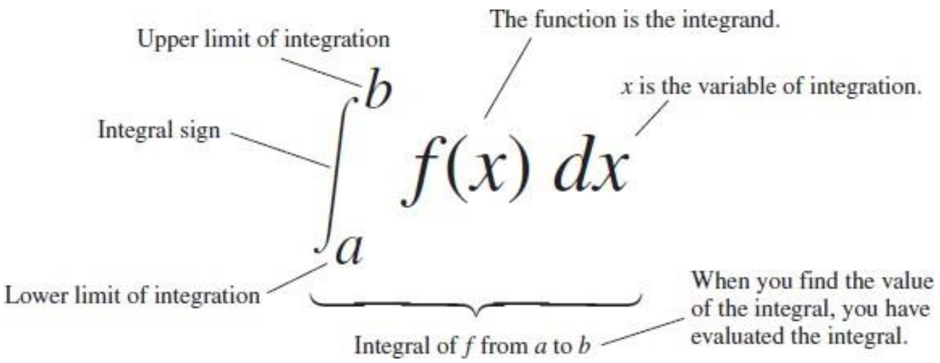
7. $\int \frac{1}{\sqrt{\frac{81}{100}-x^2}}\,dx$

8. $\int \frac{1}{\pi^2+x^2}\,dx$

9. $\int \frac{1}{\sqrt{t^2(t^2-\frac{1}{4})}}\,dx$

10. $\int \frac{1}{|x|\sqrt{x^2-7}}\,dx$

Define integration



$$\int_a^b f(x)dx = [\, F(x)\,]_a^b = F(b) - F(a)$$

Where $f(x)\, dx = F'(x)$

Rules satisfied by definite integrals		
Order of Integration:	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
Zero Width Interval:	$\int_a^a f(x)\,dx = 0$	Also a Definition
Constant Multiple:	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
	$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
Sum and Difference:	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
Additivity:	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	

Example :. Calculate the following integration.

1) $\int_2^6 \frac{dx}{x+2}$

2) $\int_{\pi/2}^{3\pi/2} \cos x\, dx$

3) $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

4) $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

5) $\int_{-2}^{4} e^{-\frac{x}{2}}\, dx$

Solution :

$$1)\ \int_{2}^{6}\frac{dx}{x+2}=ln(x+2)\Big|_{2}^{6}=ln(6+2)-ln(2+2)=ln8-ln4=3ln2-2ln2=ln2$$

$$2)\ \int_{\pi/2}^{3\pi/2}cos\,x\,dx=sin\,x\Big|_{\pi/2}^{3\pi/2}=sin(\frac{3}{2}\pi)-sin(\frac{\pi}{2})=-1-1=-2$$

$$3)\ \int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{1+x^2}=tan^{-1}\Big|_{-\sqrt{3}}^{\sqrt{3}}=tan^{-1}\sqrt{3}-tan^{-1}(-\sqrt{3})=\frac{\pi}{3}-(-\frac{\pi}{3})=\frac{2}{3}\pi$$

$$4)\ \int_{0}^{\sqrt{3}/2}\frac{dx}{\sqrt{1-x^2}}=sin^{-1}x\Big|_{0}^{\sqrt{3}/2}=sin^{-1}\frac{\sqrt{3}}{2}-sin^{-1}0=\frac{\pi}{3}-0=\frac{\pi}{3}$$

$$5)\ \int_{-2}^{4}e^{-\frac{x}{2}}dx=-2e^{-\frac{x}{2}}\Big|_{-2}^{4}=-2(e^{-2}-e)=2(e-e^{-2})$$

Example :

$$\int_0^{\frac{\pi}{2}}3\sin 2x\,dx=\left[(3)\left(-\frac{1}{2}\right)\cos 2x\right]_0^{\frac{\pi}{2}}=\left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$=\left\{-\frac{3}{2}\cos 2\left(\frac{\pi}{2}\right)\right\}-\left\{-\frac{3}{2}\cos 2(0)\right\}$$

$$=\left\{-\frac{3}{2}\cos\pi\right\}-\left\{-\frac{3}{2}\cos 0\right\}=\left\{-\frac{3}{2}(-1)\right\}-\left\{-\frac{3}{2}(1)\right\}$$

$$=\frac{3}{2}+\frac{3}{2}=\mathbf{3}$$

$$\int_1^4\left(\frac{\theta+2}{\sqrt{\theta}}\right)d\theta=\int_1^4\left(\frac{\theta}{\theta^{\frac{1}{2}}}+\frac{2}{\theta^{\frac{1}{2}}}\right)d\theta=\int_1^4\left(\theta^{\frac{1}{2}}+2\theta^{\frac{-1}{2}}\right)d\theta$$

$$=\left[\frac{\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2}+1}+\frac{2\theta^{\left(\frac{-1}{2}\right)+1}}{-\frac{1}{2}+1}\right]_1^4=\left[\frac{\theta^{\frac{3}{2}}}{\frac{3}{2}}+\frac{2\theta^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$

.

$$= \left[\frac{2}{3}\sqrt{\theta^3} + 4\sqrt{\theta}\right]_1^4 = \left\{\frac{2}{3}\sqrt{(4)^3} + 4\sqrt{4}\right\} - \left\{\frac{2}{3}\sqrt{(1)^3} + 4\sqrt{1}\right\}$$

$$= \left\{\frac{16}{3} + 8\right\} - \left\{\frac{2}{3} + 4\right\} = 5\frac{1}{3} + 8 - \frac{2}{3} - 4 = \mathbf{8\frac{2}{3}}$$

Example: Find $\int_1^9 \sqrt{5x+4}\ dx$

Solution : $\int_1^9 (5x+4)^{\frac{1}{2}}\ dx$,{ f(x) = 5x+4 , f '(x) = 5 }

$$\int_1^9 (5x+4)^{\frac{1}{2}}\ dx \times \frac{5}{5} = \frac{1}{5}\int_1^9 (5x+4)^{\frac{1}{2}}\ 5dx$$

$$= \frac{1}{5}\left[\frac{(5x+4)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^9 = \frac{1}{5}\left[\frac{(5x+4)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^9 = \frac{2}{15}\left\{(5*9+4)^{\frac{3}{2}} - (5*1+4)^{\frac{3}{2}}\right\}$$

$$= \frac{2}{15}\left\{\sqrt{49^3} - \sqrt{9^3}\right\} = \{343 - 27\} = \frac{2}{15}\{316\} = \frac{632}{15}$$

H.W.:

1. (a) $\int_1^4 5x^2\,dx$ (b) $\int_{-1}^1 -\frac{3}{4}t^2\,dt$ $\left[\text{(a) } 105 \quad \text{(b) } -\frac{1}{2}\right]$

2. (a) $\int_{-1}^2 (3-x^2)\,dx$ (b) $\int_1^3 (x^2-4x+3)\,dx$ $\left[\text{(a) } 6 \quad \text{(b) } -1\frac{1}{3}\right]$

3. (a) $\int_0^{\pi} \frac{3}{2}\cos\theta\,d\theta$ (b) $\int_0^{\frac{\pi}{2}} 4\cos\theta\,d\theta$ [(a) 0 (b) 4]

4. (a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sin 2\theta\,d\theta$ (b) $\int_0^2 3\sin t\,dt$ [(a) 1 (b) 4.248]

5. (a) $\int_0^1 5\cos 3x\,dx$ (b) $\int_0^{\frac{\pi}{6}} 3\sec^2 2x\,dx$ [(a) 0.2352 (b) 2.598]

6. (a) $\int_1^2 \csc^2 4t\,dt$ (b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (3\sin 2x - 2\cos 3x)\,dx$ [(a) 0.2572 (b) 2.638]

7. (a) $\int_0^1 3e^{3t}\,dt$ (b) $\int_{-1}^2 \frac{2}{3e^{2x}}\,dx$ [(a) 19.09 (b) 2.457]

8. (a) $\int_2^3 \frac{2}{3x}\,dx$ (b) $\int_1^3 \frac{2x^2+1}{x}\,dx$ [(a) 0.2703 (b) 9.099]

8. Integration of hyperbolic trigonometric functions

$\int \sinh x\, dx = \cosh x + C$	$\int \cosh x\, dx = \sinh x + C$
$\int sech^2 x\, dx = \tanh x + C$	$\int csch^2 x\, dx = -\coth x + C$
$\int \tanh x\, sech\, x\, dx = -sech\, x + C$	$\int csch\, x \coth x\, dx = -csch\, x + C$

Example :

1- $\int \frac{4}{\sqrt{x}} csch\sqrt{x}\ \coth\sqrt{x}dx = 4\int \frac{1}{\sqrt{x}} csch\sqrt{x} \coth\sqrt{x}dx \times \frac{2}{2}$

$= 8\int csch\sqrt{x}\ \coth\sqrt{x}\ \frac{dx}{2\sqrt{x}} = -8\, csch\sqrt{x} + C$

2- $\int \cosh^5(2x^4)\sinh(2x^4)\ x^3dx = \int \cosh^5(2x^4)\sinh(2x^4)\ x^3dx \times \frac{8}{8}$

$= \frac{1}{8}\int \cosh^5(2x^4)\sinh(2x^4)\ 8x^3dx = \frac{1}{8}\frac{\cosh^6(2x^4)}{6} + C$

$= \frac{\cosh^6(2x^4)}{48} + C$

H . W.:

1. $\int \sinh\left(\frac{x}{7}\right) dx$
2. $\int sech^2\left(x - \frac{1}{2}\right) dx$
3. $\int 6\cosh\left(\frac{x}{2} - \ln x\right)\left(\frac{1}{2} - \frac{1}{x}\right) dx$
4. $\int \frac{\cosh(\ln x)dx}{x}$
5. $\int_1^4 \frac{\cosh\sqrt{x}dx}{\sqrt{x}}$
6. $\int_0^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right) dx$

9. Integration of derivative of inverse hyperbolic trigonometric functions

$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(\frac{x}{a}) + C$	$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(\frac{x}{a}) + C$
$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + C = \frac{1}{a}\coth^{-1}(\frac{x}{a}) + C$	
$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}(\frac{x}{a}) + C$	$\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\operatorname{csch}^{-1}(\frac{x}{a}) + C$

Example :

1. $\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{x^2+4}} = [\sinh^{-1}(\frac{x}{2})]_0^{2\sqrt{3}} = \sinh^{-1}(\frac{2\sqrt{3}}{2}) - \sinh^{-1}(\frac{0}{2})$

   $= \sinh^{-1}(\sqrt{3}) - \sinh^{-1}(0) = \sinh^{-1}(\sqrt{3}) - 0 = 1.3$

2. $\int \frac{dx}{\sqrt{4x^2-3}} = \int \frac{dx}{\sqrt{(2x)^2-(\sqrt{3})^2}} = = \cosh^{-1}(\frac{x}{\sqrt{3}}) + C$

3. $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-(4x)^2}} \times \frac{4}{4} = -\frac{1}{4}\, [\operatorname{sech}^{-1}(4x)]_{1/5}^{3/13}$

$= -\frac{1}{4}\, (\operatorname{sech}^{-1}(4(3/13)) - (\operatorname{sech}^{-1}(4(1/5)))$

4. $\int_0^{\pi} \frac{\cos x\,dx}{\sqrt{1+\sin^2 x}} = [\sinh^{-1}(\sin x)]_0^{\pi} = \sinh^{-1}(\sin \pi) - \sinh^{-1}(0) = 0$

H. W.:

1. $\int_0^{1/3} \frac{6dx}{\sqrt{9x^2+1}}$
2. $\int \frac{dx}{3-x^2}$
3. $\int_1^{e} \frac{3dx}{x\sqrt{1+(\ln x)^2}}$
4. $\int_{\sqrt{2}/3}^{2/3} \frac{dy}{|y|\sqrt{9y^2-1}}$
5. $\int \frac{dx}{|x|\sqrt{5x^2-3}}$
6. $\int \frac{6dx}{\sqrt{4-25x^2}}$

Some Application of Integration

1) ***Acceleration , Velocity , Distance***

Time : t	*Acceleration :* $a(t)$
Velocity : $v(t) = \int a(t)dt$	*Distance :* $s(t) = \int v(t)dt$

Example: A body moves along a straight line according to v(t) = 2t -4 (m/s).

1. Find distance through [1,4].

$2t - 4 = 0$, $t = 2 \in [1,4]$, $s(t) = \left|\int_1^2 (2t-4)dt\right| + \left|\int_2^4 (2t-4)dt\right| = \left|[t^2-4t]_1^2\right| + \left|[t^2-4t]_2^4\right| = |-1| - |4| = 5m$

2. Find displacement through[1,4]

$s(t) = \int_1^4 (2t-4)dt = [t^2-4t]_1^4 = 3m$

3. Find distance through the tenth second.

$s(t) = \int_9^{10} (2t-4)dt = [t^2-4t]_9^{10} = \{10^2-4*10\} - \{9^2-4*9\} = 60 - 45 = 15$ m

Example: A body moves along a straight line according to the law a(t) = 4t+2 (m^2 /s). Its velocity at fourth second is 50 m/s. Determine:

1. Its velocity at any time.

$v(t) = \int a(t)dt = \int (4t+2)dt = 2t^2 + 2t + c$

$50 = 2(4)^2 + 2*4 + c$, $c = 10$, $v(t) = 2t^2 + 2t + 10$

2. Distance through [3,7].

$s(t) = \int_3^7 (2t^2 + 2t + 10)dt = [\frac{2}{3}t^3 + t^2 + 10t]_3^7 = (\frac{2}{3}(7)^3 + (7)^2 + 10*7) - (\frac{2}{3}(3)^3 + (3)^2 + 10*3) = 193.3$ m

3. Distance after 10 seconds

Note ($2t^2 + 2t + 10 \neq 0$)

$s(t) = \int_0^{10} (2t^2 + 2t + 10)dt = [\frac{2}{3}t^3 + t^2 + 10t]_0^{10} = (\frac{2}{3}(10)^3 + (10)^2 + 10*10) - (\frac{2}{3}(0)^3 + (0)^2 + 10*0) = 866.6$ m

Example: A body moves along a straight line according to the law a(t) = $21t^2 - 10$ (m/s^2). Determine its velocity after 2 seconds

$s(t) = \int_0^2 (21t^2 - 10)dt = [7t^3 - 10t]_0^2 = \{7(2)^3 - 10*2\} - \{(7(0)^3 - 10*0\} = 36$ m/s

2) Length of curve

If $f(x)$ is continuously differentiable on the interval $[a,b]$, the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Example: Find the length the curve $y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$, $0 \le x \le 1$

Solution :

$a = 0$, $b = 1$

$y' = \frac{4\sqrt{2}}{3} * \frac{3}{2} x^{1/2} = 2\sqrt{2x}$

$L = \int_0^1 \sqrt{1 + (2\sqrt{2x})^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx \quad * \frac{8}{8}$

$= \frac{1}{8} * \frac{2}{3} (1+8x)^{3/2}\Big]_0^1 = 13/6$

Example: Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 4$

Solution :

$a = 0$, $b = 4$

$y' = \frac{3}{2} x^{1/2}$

$L = \int_0^4 \sqrt{1 + (\frac{3}{2} x^{1/2})^2}\, dx = \int_0^4 \sqrt{1 + \frac{9}{4} x}\, dx \quad * \frac{9/4}{9/4}$

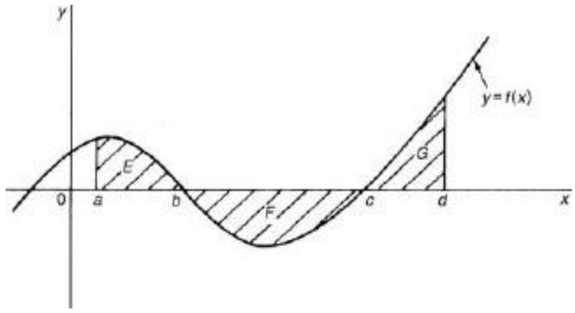
$= \frac{1}{9/4} * \frac{2}{3} [(1+\frac{9}{4} x)^{3/2}]_0^4 = \frac{8}{27} (\sqrt{10^3} - 1)$

H.W.:

Find the length of the curve $y = \frac{1}{3} x^{3/2}$ from $x = 0$ to $x = 4$

3) Areas under and between curves

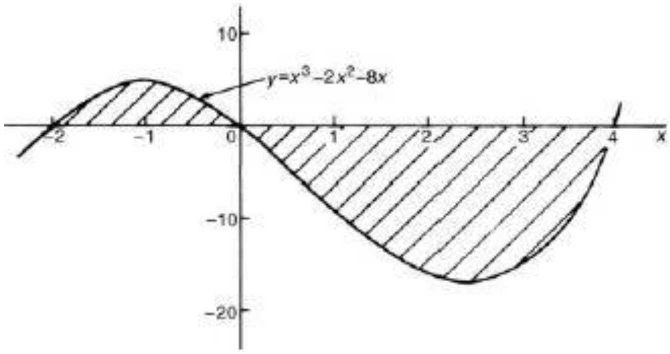
In figure below, if we need to determine the area of the function y = f(x) between the interval x = a to x = d used the following rule.



$$\text{total shaded area} = \int_a^b f(x)dx - \int_b^c f(x)dx + \int_c^d f(x)dx$$

Example 1: determine the area between the curve $y = x^3-2x^2-8x$ and the *x – axis*.

Solution : y = 0



$$x^3-2x^2-8x = 0$$
$$x(x^2-2x-8) = 0$$
$$x(x+2)(x-4) = 0$$
$$x = 0$$
$$(x+2) = 0 \qquad x = -2$$
$$(x-4) = 0 \qquad x = 4$$

Shaded area

$$= \int_{-2}^{0} (x^3 - 2x^2 - 8x)dx - \int_0^4 (x^3 - 2x^2 - 8x)dx$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_0^4$$

$$= \left(6\frac{2}{3}\right) - \left(-42\frac{2}{3}\right) = \mathbf{49\frac{1}{3}} \textbf{ square units}$$

Example: Determine the area enclosed between the curves y = $x^2 + 1$ and y = 7 – x.

Solution:

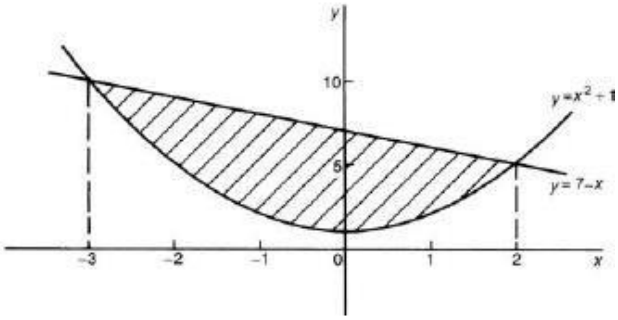
$y_1 = y_2$

$x^2 + 1 = 7 - x$

$x^2 + x - 6 = 0$

$(x-2)(x+3) = 0$

$x = 2$ and $x = -3$



$$\text{Shaded area} = \int_{-3}^{2}(7-x)\,dx - \int_{-3}^{2}(x^2+1)\,dx \quad = \int_{-3}^{2}[(7-x)-(x^2+1)]\,dx$$

$$= \int_{-3}^{2}(6-x-x^2)\,dx \quad = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2} \quad = \left(12 - 2 - \frac{8}{3}\right)$$

$$-\left(-18 - \frac{9}{2} + 9\right) \quad = \left(7\frac{1}{3}\right) - \left(-13\frac{1}{2}\right) = \mathbf{20\frac{5}{6}\ \text{square units}}$$

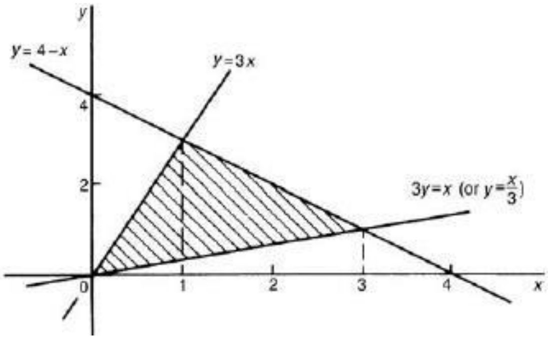
Example : Determine by integration the area bounded the three straight lines y = 4-x, y = 3x and 3y = x.

Shaded area

$$= \int_0^1 \left(3x - \frac{x}{3}\right) dx + \int_1^3 \left[(4-x) - \frac{x}{3}\right] dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^2}{6}\right]_0^1 + \left[4x - \frac{x^2}{2} - \frac{x^2}{6}\right]_1^3$$

$$= \left[\left(\frac{3}{2} - \frac{1}{6}\right) - (0)\right] + \left[\left(12 - \frac{9}{2} - \frac{9}{6}\right)\right.$$



$$\left.-\left(4 - \frac{1}{2} - \frac{1}{6}\right)\right] = \left(1\frac{1}{3}\right) + \left(6 - 3\frac{1}{3}\right) = \mathbf{4\ \text{square units}}$$

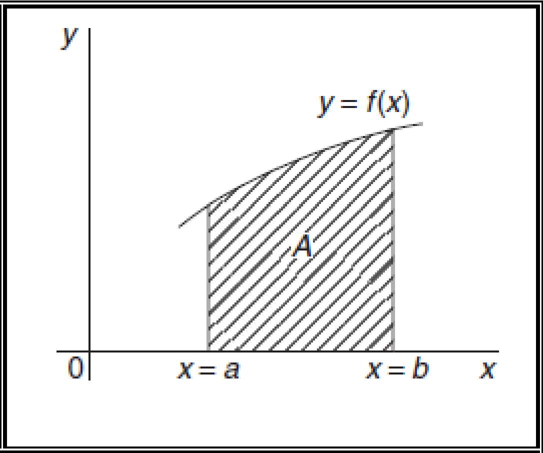
H.W.:

1. Find the area enclosed by the curve $y = 4\cos 3x$, the x-axis and ordinates $x = 0$ and $x = \dfrac{\pi}{6}$
2. Sketch the curves $y = x^2 + 3$ and $y = 7 - 3x$ and determine the area enclosed by them.
3. Determine the area enclosed by the three straight lines $y = 3x$, $2y = x$ and $y + 2x = 5$.

4) Volume of solids of revolution

The volume of revolution, V, obtained by rotating area A through one revolution about the x-axis is given by :

$$V = \int_a^b \pi y^2 \, dx$$



If a curve $x = f(y)$ is rotated 360^o about y-axis between the limits $y = c$ and $y = d$ then the volume generated , V, is given by:

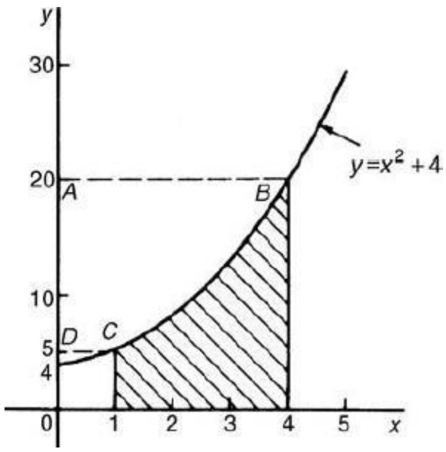
$$V = \int_c^d \pi x^2 \, dy.$$

Example: The curve $y = x^2 + 4$ is rotated one revolution about the x – axis between the limits x=1 and x=4. Determine the volume of solid of revolution produced.

Solution:

$$\text{Volume} = \int_1^4 \pi y^2 \, dx = \int_1^4 \pi (x^2 + 4)^2 \, dx$$

$$= \int_1^4 \pi (x^4 + 8x^2 + 16) \, dx = \pi \left[\frac{x^5}{5} + \frac{8x^3}{3} + 16x \right]_1^4$$



$$= \pi[(204.8 + 170.67 + 64) - (0.2 + 2.67 + 16)]$$

$$= \mathbf{420.6\pi \text{ cubic units}}$$

Example: Determine the area enclosed by the two curves $y = x^2$ and $y^2 = 8x$. if this area is rotated 360^o about the x-axis . determine the volume of the solid of revolution produced.

Solution:

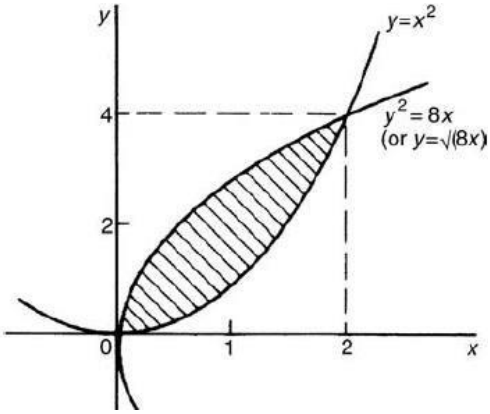
At the points of intersection the co-ordinates of the curves are equal. Since $y = x^2$ then $y^2 = x^4$. Hence equating the y^2 values at the points of intersection:

$$x^4 = 8x$$

from which, $x^4 - 8x = 0$

and $x(x^3 - 8) = 0$

Hence, at the points of intersection, x =0 and x =2. When x= 0 , y =0 and when x= 2, y =4.



The points of intersection of the curves $y = x^2$ and $y^2 = 8x$ are therefore at (0,0) and (2,4). If $y^2 = 8x$ then $y = \sqrt{8x}$

Shaded area

$$= \int_0^2 \left(\sqrt{8x} - x^2\right) dx = \int_0^2 \left(\sqrt{8}\right) x^{\frac{1}{2}} - x^2 \Big) dx = \left[\left(\sqrt{8}\right)\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^2 = \left\{\frac{\sqrt{8}\sqrt{8}}{\frac{3}{2}} - \frac{8}{3}\right\} - \{0\}$$

$$= \frac{16}{3} - \frac{8}{3} = \frac{8}{3} = 2\frac{2}{3} \textbf{square units}$$

The volume produced by revolving the shaded area about the x-axis is given by:

{(volume produced by revolving $y^2 = 8x$)

− (volume produced by revolving $y = x^2$)}

$$\textbf{volume} = \int_0^2 \pi(8x)dx - \int_0^2 \pi(x^4)dx$$

$$= \pi\int_0^2 (8x - x^4)dx = \pi\left[\frac{8x^2}{2} - \frac{x^5}{5}\right]_0^2$$

$$= \pi\left[\left(16 - \frac{32}{5}\right) - (0)\right]$$

$$= \mathbf{9.6\pi} \textbf{ cubic units}$$

H.W.:

1. The curve $xy = 3$ is revolved one revolution about the x-axis between the limits $x = 2$ and $x = 3$. Determine the volume of the solid produced.
2. The area between $\frac{y}{x^2} = 1$ and $y + x^2 = 8$ is rotated 360° about the x-axis. Find the volume produced.
3. The curve $y = 2x^2 + 3$ is rotated about (a) the x-axis between the limits $x = 0$ and $x = 3$, and (b) the y-axis, between the same limits. Determine the volume generated in each case.