MAXIMA, MINIMA AND SADDLE POINTS FOR FUNCTIONS OF TWO VARIABLES

1. Find the stationary point of the surface $f(x, y) = x^2 + y^2$ and determine its nature. Sketch the contour map represented by z.

Let $f(x, y) = z = x^2 + y^2$

(i) $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 2y$

(ii) For stationary points, $2x = 0$ from which, $x = 0$

and $2y = 0$ from which, $y = 0$

(iii) The coordinates of the stationary point are (0, 0)

(iv) $\frac{\partial^2 z}{\partial x^2} = 2 \quad \frac{\partial^2 z}{\partial y^2} = 2 \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(2y) = 0$

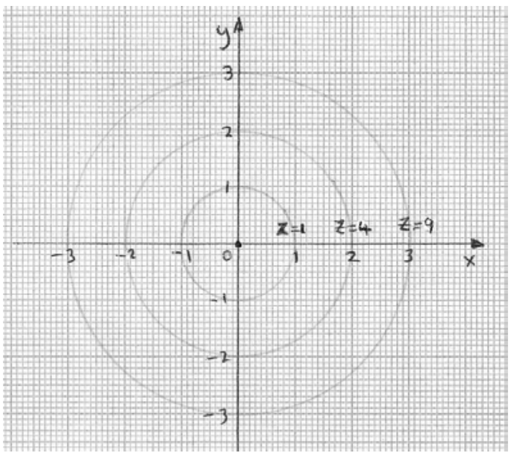
(v) When $x = 0$, $y = 0$, $\frac{\partial^2 z}{\partial x^2} = 2 \quad \frac{\partial^2 z}{\partial y^2} = 2$ and $\frac{\partial^2 z}{\partial x \partial y} = 0$

(vi) $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

(vii) $\Delta_{(0,0)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (2)(2) = -4$ which is negative

(viii) Since $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$ then (0, 0) is a minimum point.

The contour map for z is shown below.



2. Find the maxima, minima and saddle points for the following functions:

(a) $f(x, y) = x^2 + y^2 - 2x + 4y + 8$

(b) $f(x, y) = x^2 - y^2 - 2x + 4y + 8$

(c) $f(x, y) = 2x + 2y - 2xy - 2x^2 - y^2 + 4$

(a) Let $f(x, y) = z = x^2 + y^2 - 2x + 4y + 8$

(i) $\dfrac{\partial z}{\partial x} = 2x - 2$ and $\dfrac{\partial z}{\partial y} = 2y + 4$

(ii) For stationary points, $2x - 2 = 0$ from which, $x = 1$

and $2y + 4 = 0$ from which, $y = -2$

(iii) The coordinates of the stationary point are (1, –2)

(iv) $\dfrac{\partial^2 z}{\partial x^2} = 2 \qquad \dfrac{\partial^2 z}{\partial y^2} = 2 \qquad \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}(2y + 4) = 0$

(v) When $x = 1$, $y = -2$, $\dfrac{\partial^2 z}{\partial x^2} = 2 \qquad \dfrac{\partial^2 z}{\partial y^2} = 2$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

(vi) $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

(vii) $\Delta_{(1,-2)} = \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (2)(2) = -4$ which is negative

(viii) Since $\Delta < 0$ and $\dfrac{\partial^2 z}{\partial x^2} > 0$ then (1, –2) is a minimum point.

(b) Let $f(x, y) = z = x^2 - y^2 - 2x + 4y + 8$

(i) $\dfrac{\partial z}{\partial x} = 2x - 2$ and $\dfrac{\partial z}{\partial y} = -2y + 4$

(ii) For stationary points, $2x - 2 = 0$ from which, $x = 1$

and $-2y + 4 = 0$ from which, $y = 2$

(iii) The coordinates of the stationary point are (1, 2)

(iv) $\dfrac{\partial^2 z}{\partial x^2} = 2 \qquad \dfrac{\partial^2 z}{\partial y^2} = -2 \qquad \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}(-2y + 4) = 0$

(v) When $x = 1$, $y = 2$, $\dfrac{\partial^2 z}{\partial x^2} = 2 \qquad \dfrac{\partial^2 z}{\partial y^2} = -2$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

(vi) $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

(vii) $\Delta_{(1,2)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (2)(-2) = 4$ which is positive

(viii) Since $\Delta > 0$ then (1, 2) is a saddle point

(c) Let $f(x, y) = z = 2x + 2y - 2xy - 2x^2 - y^2 + 4$

(i) $\frac{\partial z}{\partial x} = 2 - 2y - 4x$ and $\frac{\partial z}{\partial y} = 2 - 2x - 2y$

(ii) For stationary points, $2 - 2y - 4x = 0$

i.e. $1 - y - 2x = 0$ (1)

and $2 - 2x - 2y = 0$

i.e. $1 - x - y = 0$ (2)

From (1), $y = 1 - 2x$

Substituting in (2) gives: $1 - x -(1 - 2x) = 0$

i.e. $1 - x - 1 + 2x = 0$ from which, $x = 0$

When $x = 0$ in equations (1) and (2), $y = 1$

(iii) The coordinates of the stationary point are (0, 1)

(iv) $\frac{\partial^2 z}{\partial x^2} = -4$ $\frac{\partial^2 z}{\partial y^2} = -2$ $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(2 - 2x - 2y) = -2$

(v) When $x = 0$, $y = 1$, $\frac{\partial^2 z}{\partial x^2} = -4$ $\frac{\partial^2 z}{\partial y^2} = -2$ and $\frac{\partial^2 z}{\partial x \partial y} = -2$

(vi) $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (-2)^2 = 4$

(vii) $\Delta_{(0,1)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = 4 - (-4)(-2) = -4$ which is negative

(viii) Since $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$ then (0, 1) is a maximum point

3. Determine the stationary values of the function $f(x, y) = x^3 - 6x^2 - 8y^2$ and distinguish between them.

Let $f(x, y) = z = x^3 - 6x^2 - 8y^2$

(i) $\frac{\partial z}{\partial x} = 3x^2 - 12x$ and $\frac{\partial z}{\partial y} = -16y$

(ii) For stationary points, $3x^2 - 12x = 0$ i.e. $3x(x - 4)$ from which, $x = 0$ and $x = 4$

and $-16y = 0$ from which, $y = 0$

(iii) The coordinates of the stationary points are (0, 0) and (4, 0)

(iv) $\frac{\partial^2 z}{\partial x^2} = 6x - 12$ $\frac{\partial^2 z}{\partial y^2} = -16$ $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(-16y) = 0$

(v) When $x = 0$, $y = 0$, $\frac{\partial^2 z}{\partial x^2} = -12$ $\frac{\partial^2 z}{\partial y^2} = -16$ and $\frac{\partial^2 z}{\partial x \partial y} = 0$

When $x = 4$, $y = 0$, $\frac{\partial^2 z}{\partial x^2} = 12$ $\frac{\partial^2 z}{\partial y^2} = -16$ and $\frac{\partial^2 z}{\partial x \partial y} = 0$

(vi) For (0, 0), $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

For (4, 0), $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

(vii) $\Delta_{(0,0)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (-12)(-16) = -192$ which is negative

$\Delta_{(4,0)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (12)(-16) = +192$ which is positive

(viii) Since $\Delta_{(0,0)} < 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$ then (0, 0) is a maximum point

Since $\Delta_{(4,0)} > 0$ then (4, 0) is a saddle point

4. Locate the stationary points of the function $z = 12x^2 + 6xy + 15y^2$

(i) $\frac{\partial z}{\partial x} = 24x + 6y$ and $\frac{\partial z}{\partial y} = 6x + 30y$

(ii) For stationary points, $24x + 6y = 0$ (1)

and $6x + 30y = 0$ (2)

(iii) From (1), $6y = -24x$ i.e. $y = -4x$

Substituting in (2) gives: $6x + 30(-4x) = 0$

i.e. $6x = 120x$ i.e. $x = 0$

When $x = 0$, $y = 0$, hence the coordinates of the stationary point are (0, 0)

(iv) $\frac{\partial^2 z}{\partial x^2} = 24$ $\frac{\partial^2 z}{\partial y^2} = 30$ $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(6x + 30y) = 6$

(v) When $x = 0$, $y = 0$, $\frac{\partial^2 z}{\partial x^2} = 24$ $\frac{\partial^2 z}{\partial y^2} = 30$ and $\frac{\partial^2 z}{\partial x \partial y} = 6$

(vi) $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 36$

(vii) $\Delta_{(0,0)} = \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) = (6)^2 - (24)(30)$ which is negative

(viii) Since $\Delta < 0$ and $\dfrac{\partial^2 z}{\partial x^2} > 0$ then (0, 0) is a minimum point

5. Find the stationary points of the surface $z = x^3 - xy + y^3$ and distinguish between them.

(i) $\dfrac{\partial z}{\partial x} = 3x^2 - y$ and $\dfrac{\partial z}{\partial y} = -x + 3y^2$

(ii) For stationary points, $3x^2 - y = 0$ (1)

and $-x + 3y^2 = 0$ (2)

(iii) From (1), $y = 3x^2$

Substituting in (2) gives: $-x + 3(3x^2)^2 = 0$

$-x + 27x^4 = 0$

and $x(27x^3 - 1) = 0$

i.e. $x = 0$ or $27x^3 - 1 = 0$ i.e. $x^3 = \dfrac{1}{27}$ and $x = \sqrt[3]{\left(\dfrac{1}{27}\right)} = \dfrac{1}{3}$

Hence, $x = 0$ or $x = \dfrac{1}{3}$

From (1), when $x = 0$, $y = 0$

and when $x = \dfrac{1}{3}$, $y = 3x^2 = 3\left(\dfrac{1}{3}\right)^2 = \dfrac{1}{3}$

Hence, the stationary points occur at (0, 0) and $\left(\dfrac{1}{3}, \dfrac{1}{3}\right)$

(iv) $\dfrac{\partial^2 z}{\partial x^2} = 6x$ $\dfrac{\partial^2 z}{\partial y^2} = 6y$ $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}(-x + 3y^2) = -1$

(v) For (0, 0), $\dfrac{\partial^2 z}{\partial x^2} = 0$ $\dfrac{\partial^2 z}{\partial y^2} = 0$ and $\dfrac{\partial^2 z}{\partial x \partial y} = -1$

For $\left(\dfrac{1}{3}, \dfrac{1}{3}\right)$, $\dfrac{\partial^2 z}{\partial x^2} = 2$ $\dfrac{\partial^2 z}{\partial y^2} = 2$ and $\dfrac{\partial^2 z}{\partial x \partial y} = -1$

(vi) For (0, 0), $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 1$

For $\left(\dfrac{1}{3}, \dfrac{1}{3}\right)$, $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 1$

(vii) $\Delta_{(0,0)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = 1-(0)(0) = 1$ which is positive

$\Delta_{\left(\frac{1}{3},\frac{1}{3}\right)} = 1-(2)(2) = -3$ which is negative

(viii) Since $\Delta_{(0,0)} > 0$ then (0, 0) is a saddle point

$\Delta_{\left(\frac{1}{3},\frac{1}{3}\right)} < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$ then $\left(\frac{1}{3}, \frac{1}{3}\right)$ is a minimum point

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EXERCISE

1. The function $z = x^2 + y^2 + xy + 4x - 4y + 3$ has one stationary value. Determine its coordinates and its nature.

(i) $\dfrac{\partial z}{\partial x} = 2x + y + 4$ and $\dfrac{\partial z}{\partial y} = 2y + x - 4$

(ii) For stationary points, $2x + y + 4 = 0$ (1)

and $2y + x - 4 = 0$ (2)

(iii) (1) + (2) gives: $3x + 3y = 0$ from which, $y = -x$

Substituting in (1), $2x - x + 4 = 0$ i.e. $x = -4$, thus $y = +4$

Hence, the stationary point occurs at (–4, 4)

(iv) $\dfrac{\partial^2 z}{\partial x^2} = 2$ $\dfrac{\partial^2 z}{\partial y^2} = 2$ $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}(2y + x - 4) = 1$

(v) When $x = -4$, $y = 4$, $\dfrac{\partial^2 z}{\partial x^2} = 2$ $\dfrac{\partial^2 z}{\partial y^2} = 2$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 1$

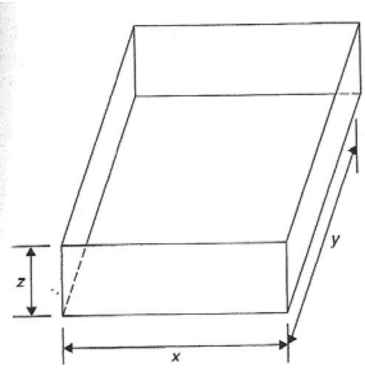
(vi) $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 1$

(vii) $\Delta_{(-4,4)} = \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) = (1)^2 - (2)(2) = -3$ which is negative

(viii) Since $\Delta < 0$ and $\dfrac{\partial^2 z}{\partial x^2} > 0$ then (–4, 4) is a minimum point

2. An open rectangular container is to have a volume of 32 m^3. Determine the dimensions and the total surface area such that the total surface area is a minimum.

Let the dimensions of the container be x, y and z as shown below.



Volume $V = xyz = 32$ (1)

Surface area, $S = xy + 2yz + 2xz$ (2)

From equation (1), $z = \frac{32}{xy}$

Substituting in equation (2) gives:

$$S = xy + 2y\left(\frac{32}{xy}\right) + 2x\left(\frac{32}{xy}\right)$$

i.e. $S = xy + \frac{64}{x} + \frac{64}{y}$ which is a function of two variables

$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0$ for a stationary point, hence $x^2y = 64$ (3)

$\frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0$ for a stationary point, hence $xy^2 = 64$ (4)

Dividing equation (3) by (4) gives:

$$\frac{x^2y}{xy^2} = 1 \quad \text{i.e.} \quad \frac{x}{y} = 1 \quad \text{i.e.} \quad x = y$$

Substituting $y = x$ in equation (3) gives $x^3 = 64$, from which, $x = 4$ m

Hence $y = 4$ m also

From equation (1), $(4)(4)z = 32$ from which, $z = \frac{32}{16} = 2$ m

$$\frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}, \quad \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3} \quad \text{and} \quad \frac{\partial^2 S}{\partial x \partial y} = 1$$

When $x = y = 4$, $\frac{\partial^2 S}{\partial x^2} = 2$, $\frac{\partial^2 S}{\partial y^2} = 2$ and $\frac{\partial^2 S}{\partial x \partial y} = 1$

$$\Delta = (1)^2 - (2)(2) = -3$$

Since $\Delta < 0$ and $\frac{\partial^2 S}{\partial x^2} > 0$, then the surface area S is a minimum

Hence the minimum dimensions of the container to have a volume of 32 m^3 are 4 m by 4 m by 2 m

From equation (2), minimum surface area, $S = (4)(4) + 2(4)(2) + 2(4)(2)$

$= 16 + 16 + 16 = 48\ m^2$

3. Determine the stationary values of the function $f(x, y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$ and distinguish between them.

Let $f(x, y) = z = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$

(i) $\frac{\partial z}{\partial x} = 4x^3 + 8xy^2 - 4x$ and $\frac{\partial z}{\partial y} = 8x^2y + 4y$

(ii) For stationary points, $4x^3 + 8xy^2 - 4x = 0$ (1)

and $8x^2y + 4y = 0$ (2)

(iii) From (2), $4y(2x^2 - 1) = 0$ from which, $y = 0$

From (1), if $y = 0$, $4x^3 - 4x = 0$ i.e. $4x(x^2 - 1) = 0$

from which, $x = 0$ or $x = \pm 1$

Hence, the stationary points occur at (0, 0) and (1, 0) and (–1, 0)

(iv) $\frac{\partial^2 z}{\partial x^2} = 12x^2 + 8y^2 - 4$ $\frac{\partial^2 z}{\partial y^2} = 8x^2 + 4$ $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(8x^2y + 4y) = 16xy$

(v) For (0, 0), $\frac{\partial^2 z}{\partial x^2} = -4$ $\frac{\partial^2 z}{\partial y^2} = 4$ and $\frac{\partial^2 z}{\partial x \partial y} = 0$

For (1, 0), $\frac{\partial^2 z}{\partial x^2} = 8$ $\frac{\partial^2 z}{\partial y^2} = 12$ and $\frac{\partial^2 z}{\partial x \partial y} = 0$

For (–1, 0), $\frac{\partial^2 z}{\partial x^2} = 8$ $\frac{\partial^2 z}{\partial y^2} = 12$ and $\frac{\partial^2 z}{\partial x \partial y} = 0$

(vi) For all three stationary points, $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

(vii) $\Delta_{(0,0)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (-4)(4) = 32$ which is positive

$\Delta_{(1,0)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (8)(12) = -96$ which is negative

$\Delta_{(-1,0)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (8)(12) = -96$ which is negative

(viii) Since $\Delta_{(0,0)} > 0$, then (0, 0) is a saddle point

Since $\Delta_{(1,0)} < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$ then (1, 0) is a minimum point

Since $\Delta_{(-1,0)} < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$ then (– 1, 0) is a minimum point

4. Determine the stationary points of the surface $f(x, y) = x^3 - 6x^2 - y^2$

Let $f(x, y) = z = x^3 - 6x^2 - y^2$

(i) $\frac{\partial z}{\partial x} = 3x^2 - 12x$ and $\frac{\partial z}{\partial y} = -2y$

(ii) For stationary points, $3x^2 - 12x = 0$ (1)

and $-2y = 0$ (2)

(iii) From (1) $3x(x-4) = 0$ from which, $x = 0$ or $x = 4$

From (2), $y = 0$

Hence, the stationary points occur at (0, 0) and (4, 0)

(iv) $\frac{\partial^2 z}{\partial x^2} = 6x - 12$ $\frac{\partial^2 z}{\partial y^2} = -2$ $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(-2y) = 0$

(v) At (0, 0), $\frac{\partial^2 z}{\partial x^2} = -12$ $\frac{\partial^2 z}{\partial y^2} = -2$ and $\frac{\partial^2 z}{\partial x \partial y} = 0$

At (4, 0), $\frac{\partial^2 z}{\partial x^2} = 12$ $\frac{\partial^2 z}{\partial y^2} = -2$ and $\frac{\partial^2 z}{\partial x \partial y} = 0$

(vi) For both points, $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

(vii) $\Delta_{(0,0)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (-12)(-2) = -24$ which is negative

$\Delta_{(4,0)} = (0)^2 - (12)(-2) = 24$ which is positive

(viii) Since $\Delta_{(0,0)} < 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$ then (0, 0) is a maximum point

Since $\Delta_{(4,0)} > 0$ then (4, 0) is a saddle point

5. Locate the stationary points on the surface

$f(x, y) = 2x^3 + 2y^3 - 6x - 24y + 16$

and determine their nature.

Let $f(x, y) = z = 2x^3 + 2y^3 - 6x - 24y + 16$

(i) $\dfrac{\partial z}{\partial x} = 6x^2 - 6$ and $\dfrac{\partial z}{\partial y} = 6y^2 - 24$

(ii) For stationary points, $6x^2 - 6 = 0$ (1)

and $6y^2 - 24 = 0$ (2)

(iii) From (1), $6x^2 = 6$ from which, $x = \pm 1$

From (2), $6y^2 = 24$ i.e. $y^2 = 4$ and $y = \pm 2$

Hence, the stationary points occur at (1, 2) and (–1, –2) and (–1, 2) and (1, –2)

(iv) $\dfrac{\partial^2 z}{\partial x^2} = 12x$ $\dfrac{\partial^2 z}{\partial y^2} = 12y$ $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(6y^2 - 24\right) = 0$

(v) For (1, 2), $\dfrac{\partial^2 z}{\partial x^2} = 12$ $\dfrac{\partial^2 z}{\partial y^2} = 24$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

For (–1, –2), $\dfrac{\partial^2 z}{\partial x^2} = -12$ $\dfrac{\partial^2 z}{\partial y^2} = -24$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

For (–1, 2), $\dfrac{\partial^2 z}{\partial x^2} = -12$ $\dfrac{\partial^2 z}{\partial y^2} = 24$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

For (1, –2), $\dfrac{\partial^2 z}{\partial x^2} = 12$ $\dfrac{\partial^2 z}{\partial y^2} = -24$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

(vi) For all four stationary points, $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

(vii) $\Delta_{(1,2)} = \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (12)(24) = -288$ which is negative

$\Delta_{(-1,-2)} = \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (-12)(-24) = -288$ which is negative

$\Delta_{(-1,2)} = \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (-12)(24) = +288$ which is positive

$\Delta_{(1,-2)} = \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) = (0)^2 - (12)(-24) = +288$ which is positive

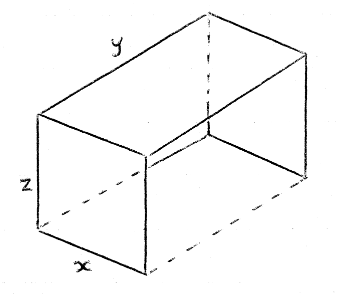
(viii) Since $\Delta_{(-1,2)} > 0$, then (–1, 2) is a saddle point

Since $\Delta_{(1,-2)} > 0$, then (1, –2) is a saddle point

Since $\Delta_{(1,2)} < 0$ and $\dfrac{\partial^2 z}{\partial x^2} > 0$ then (1, 2) is a minimum point

Since $\Delta_{(-1,-2)} < 0$ and $\dfrac{\partial^2 z}{\partial x^2} < 0$ then (–1, –2) is a maximum point

6. A large marquee is to be made in the form of a rectangular box-like shape with canvas covering on the top, back and sides. Determine the minimum surface area of canvas necessary if the volume of the marquee is to be 250 m^3.



A sketch of the marquee is shown above

Volume of marquee, $V = xyz = 250$ (1)

Surface area, $S = xy + yz + 2xz$ (2)

From (1), $z = \frac{250}{xy}$

Substituting in (2) gives: $S = xy + y\left(\frac{250}{xy}\right) + 2x\left(\frac{250}{xy}\right) = xy + \frac{250}{x} + \frac{500}{y} = xy + 250x^{-1} + 500y^{-1}$

$\frac{\partial S}{\partial x} = y - \frac{250}{x^2}$ and $\frac{\partial S}{\partial y} = x - \frac{500}{y^2}$

For a stationary point, $\frac{\partial S}{\partial x} = y - \frac{250}{x^2} = 0$

from which, $y = \frac{250}{x^2}$ or $yx^2 = 250$ (3)

and $\frac{\partial S}{\partial y} = x - \frac{500}{y^2} = 0$

from which, $x = \frac{500}{y^2}$ or $xy^2 = 500$ (4)

Dividing equation (3) by equation (4) gives:

$\frac{yx^2}{xy^2} = \frac{250}{500}$ i.e. $\frac{x}{y} = \frac{1}{2}$ and $y = 2x$

Substituting $y = 2x$ in equation (3) gives: $2x^3 = 250$ and $x = \sqrt[3]{125} = 5$ m

and $y = 2x = 10$ m

From equation (1), $xyz = 250$ i.e. $(5)(10)z = 250$ from which, $z = 5$ m

$$\frac{\partial^2 S}{\partial x^2} = \frac{750}{x^3} \qquad \frac{\partial^2 S}{\partial y^2} = \frac{1000}{y^3} \qquad \text{and} \qquad \frac{\partial^2 S}{\partial x \partial y} = 1$$

When $x = 5$ and $y = 10$, $\quad \frac{\partial^2 S}{\partial x^2} = 6 \qquad \frac{\partial^2 S}{\partial y^2} = 1 \qquad$ and $\quad \frac{\partial^2 S}{\partial x \partial y} = 1$

$$\Delta = \left(\frac{\partial^2 S}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 S}{\partial x^2}\right)\left(\frac{\partial^2 S}{\partial y^2}\right) = (1)^2 - (6)(1) = -6 \quad \text{which is negative}$$

Since $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$ then the surface area is a minimum.

Minimum surface area, $S = xy + yz + 2xz$

$= (5)(10) + (10)(5) + (2)(5)(5)$

$= 50 + 50 + 50 = 150 \text{ m}^2$
