Numerical Methods

Runge - kutta Method (4^{th} order)

$\frac{dy}{dx} = f(x, y)$ with x_0 & y_0

$$\boxed{y_1 = y_0 + k}$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h \cdot f(x_0, y_0)$$

$$k_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$$

Ex1/ Solve $\frac{dy}{dx} = 3x + y^2$ with $x_0 = 1$, $y_0 = 1.2$ by using Runge-kutta Method of 4th order at $x = 1.1$

Sol'y $f(x,y) = 3x + y^2$, $x_0 = 1$, $y_0 = 1.2$

$x_1 = x_0 + h$

$1.1 = 1 + h$

$\boxed{h = 0.1}$

$k_1 = h \cdot f(x_0, y_0)$

$= (0.1)\left[3(1) + (1.2)^2\right] = \boxed{0.444}$

$k_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= (0.1)\left[3\left(1 + \frac{0.1}{2}\right) + \left(1.2 + \frac{0.444}{2}\right)^2\right] = \boxed{0.5172}$

$k_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= (0.1)\left[3\left(1 + \frac{0.1}{2}\right) + \left(1.2 + \frac{0.5172}{2}\right)^2\right] = 0.52775$

$= \boxed{0.5278}$

$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$

$= (0.1)\left[3(1+0.1) + (1.2 + 0.5278)^2\right] = \boxed{0.6285}$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$ (3)

$$= \frac{1}{6}(0.444 + 2(0.5172) + 2(0.5278) + 0.6285)$$

$$\boxed{k = 0.5271}$$

$$y = y_0 + k$$

$$= 1.2 + 0.5271$$

$$\boxed{y = 1.7271} \quad \text{at } x = 1.1$$

EX2/ Solve $\frac{dy}{dx} = \frac{1}{x+y}$ with $x_0 = 0$, $y_0 = 2$ at $x = 0.4$ taking $h = 0.2$ by Runge-Kutta Method of 4th order.

Sol/ $f(x,y) = \frac{1}{x+y}$, $x_0 = 0$, $y_0 = 2$, $h = 0.2$

1st iteration

$k_1 = h \cdot f(x_0, y_0)$

$= (0.2) \cdot \left(\frac{1}{0+2}\right) = \boxed{0.1}$

$k_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= (0.2) \cdot \left(\frac{1}{0 + \frac{0.2}{2} + 2 + \frac{0.1}{2}}\right) = \boxed{0.093}$

$k_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.2)\left(\frac{1}{0 + \frac{0.2}{2} + 2 + \frac{0.093}{2}}\right) = \boxed{0.0932}$

$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$

$= (0.2)\left[\frac{1}{0+0.2+2+0.0932}\right] = \boxed{0.0872}$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \boxed{0.0933}$

$y_1 = y_0 + k$

$= 2 + 0.0933 = \boxed{2.0933}$ at $\boxed{x_1 = 0.2}$

2nd iteration

$x_1 = 0.2$, $y_1 = 2.0933$, $h = 0.2$

$$k_1 = h \cdot f(x_1, y_1)$$

$$= (0.2)\left[\frac{1}{0.2 + 2.0933}\right] = \boxed{0.0872}$$

$$k_2 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.2)\left[\frac{1}{0.2 + \frac{0.2}{2} + 2.0933 + \frac{0.0872}{2}}\right] = \boxed{0.0821}$$

$$k_3 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.2)\left[\frac{1}{0.2 + \frac{0.2}{2} + 2.0933 + \frac{0.0821}{2}}\right] = \boxed{0.0822}$$

$$k_4 = h \cdot f(x_1 + h, y_1 + k_3)$$

$$= (0.2)\left[\frac{1}{0.2 + 0.2 + 2.0933 + 0.0822}\right] = \boxed{0.0777}$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \boxed{0.08225}$$

$$y_2 = y_1 + k$$

$$= 2.0933 + 0.08225$$

$\boxed{y_2 = 2.1775}$ Ans. at $\boxed{x = 0.4}$