Chapter Three

Hydrostatic Forces on Surfaces . Dr.Abdulkareem Abdulwahab

3.1/ Vertical Plane Surface Submerged in Liquid :

Consider a plane vertical surface of arbitrary shape immersed in a liquid as shown in Fig.(3.1) :

Let , A – Total area of the Surface. h_c - Distance of center of gravity(C.G)of the area from free surface of liquid. G – Center of gravity of plane surface. P – Center Of pressure. h_p – Distance of center of pressure from free surface of liquid.

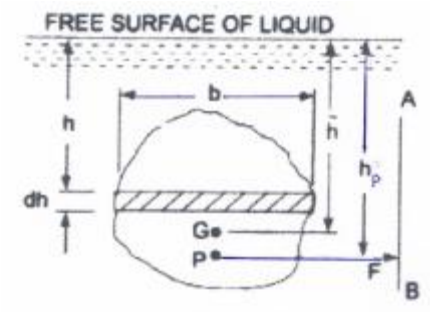


Fig.(3.1)

Consider a strip of thickness (dh) and width(b) at a depth of (h) from free surface of liquid as shown in Fig.(3.1) :

Pressure on the strip $P = \rho\, g\, h$

Area of the strip $A = b \times dh$

Total Force on strip $dF = p \times \text{Area} = \rho\, g\, h \times b \times dh$

Total Force on the whole surface $F = \int dF = \int \rho\, gh \times b \times dh$

$$= \rho\, g \int b \times h \times dh$$

$$\int b \times h \times dh = \int h \times dA$$

= Moment of surface area about the free surface of liquid.

= Area of surface × Distance of (C.G) from free surface .

$$= A \times h_c$$

$$F = \rho\, g\, A\, h_c \qquad (3.1)$$

<u>Center of pressure (p)</u> : Center of pressure is calculated by using the (principle of moments) , which states that the moment of the resultant force about an axis is equal to the sum of moments of the components about the same axis . (The distance of center of pressure (p) from the free surface is h_p).

Moment of Force = dF × h

$$= \rho\, g\, h \times b \times dh \times h \qquad (3.2)$$

Sum of momentum of all such forces = $\int \rho\, g\, h \times b \times dh \times h$

$$= \rho\, g \int b h^2\, dh = \rho\, g \int h^2\, dA$$

$$= \rho\, g \int b h^2\, dh = \rho\, g\, I_o \qquad (3.3)$$

(In which I_o is moment of inertia of the surface about free surface of liquid)

But , the moment of the force F about free surface of the liquid = $F \times h_p$

Therefore , $F \times h_p = \rho\, g\, I_o$

But , $F = \rho\, g\, A\, h_c$

Therefore , $\rho\, g\, A\, h_c \times h_p = \rho\, g\, I_o$

$$h_p = \frac{\rho\, g\, I_o}{\rho\, g\, A\, h_c} = \frac{I_o}{A\, h_c} \qquad (3.4)$$

By the theorem of parallel axis , we have

$$I_o = I_G + A \times h_c^2$$

Where I_G = Moment of Inertia of area about an axis passing through the C.G of the area and parallel to the free surface of liquid.

Substituting I_o in equation (3.4) , we get ,

$$h_p = \frac{I_G + A\, h_c^2}{A\, h_c} = \frac{I_G}{A\, h_c} + h_c \qquad (3.5)$$

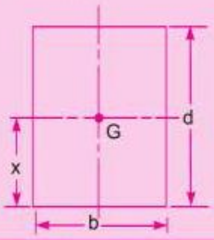
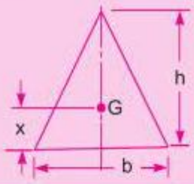
The center of pressure h_p lies below the center of gravity of the vertical surface h_c .

3.2 / <u>Horizontal plane surface submerged in liquid</u> :

Consider a plane horizontal surface immersed in a static fluid. As every point of he surface is at the same depth from the free surface of the liquid, the pressure will be equal on the entire surface and equal to :

$P = \rho g h$ (where h is depth of the surface)

Table (3.1) The moments of inertia and other geometric properties of some important plane surfaces .

Plane surface	*C.G. from the base*	*Area*	*Moment of inertia about an axis passing through C.G. and parallel to base (I_G)*	*Moment of inertia about base (I_0)*
1. Rectangle 	$x = \frac{d}{2}$	bd	$\frac{bd^3}{12}$	$\frac{bd^3}{3}$
2. Triangle 	$x = \frac{h}{3}$	$\frac{bh}{2}$	$\frac{bh^3}{36}$	$\frac{bh^3}{12}$

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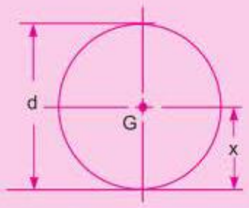
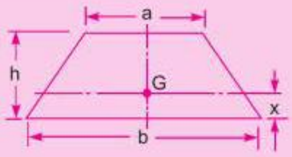
Plane surface	*C.G. from the base*	*Area*	*Moment of inertia about an axis passing through C.G. and parallel to base (I_G)*	*Moment of inertia about base (I_0)*
3. Circle 	$x = \frac{d}{2}$	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$	—
4. Trapezium 	$x = \left(\frac{2a+b}{a+b}\right)\frac{h}{3}$	$\frac{(a+b)}{2} \times h$	$\left(\frac{a^2+4ab+b^2}{36(a+b)}\right) \times h^3$	—

3.3 / Inclined Plane surface submerged in liquid :

Consider a plane surface of arbitrary shape immersed in a liquid in such a way that the plane of the surface makes an angle ϴ with the free surface of the liquid as shown in Fig.(3.2).

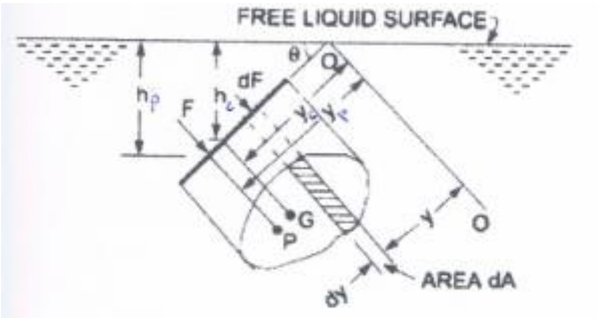


Fig.(3.2) Inclined immersed surface

Let , A total area of inclined surface , h_c depth of C.G of inclined area from free surface , h_p distance of center of pressure from free surface of liquid , ϴ angle made by the plane of the surface with free surface , y_c distance of the C.G of the inclined surface from O – O , y_p distance of the center of pressure from the O – O .

Consider a small strip of area dA at a depth (h) from free surface and at a distance y from the axis O – O as shown in Fig.(3.2).

Force dF on the strip = p × Area of strip = ρ g h ×dA

Total Force on the whole area , F = ∫ dF = ∫ ρ g h dA

But from Fig.(3.2) , $\sin\Theta = \frac{h}{y} = \frac{h_c}{y_c} = \frac{h_p}{y_p}$

Therefore , h = y sin ϴ

$$F = \int \rho\, g \times y \sin\Theta \times dA = \rho\, g \sin\Theta \int y\, dA$$

But , $\int y\, dA = A\, y_c$

Therefore , $F = \rho\, g \sin\Theta \times A \times y_c$

$$F = \rho\, g\, A\, h_c \qquad (3.6)$$

Force on the strip , dF = ρ g h dA

$\sin\Theta = \frac{h}{y}$, $h = y \sin\Theta$

$$dF = \rho\, g\, y \sin\Theta\, dA$$

Moment of force (dF) about axis O – O ,

$$dF \times y = \rho\, g\, y \sin\Theta\, dA \times y = \rho\, g \sin\Theta\, y^2\, dA$$

Sum of moments of all such forces about O - O ,

$$M = \int \rho\, g \sin\Theta\, y^2\, dA = \rho\, g \sin\Theta \int y^2\, dA$$

But $\int y^2\, dA = I_o$

Therefore , $M = \rho\, g \sin\Theta\, I_o$ **(3.7)**

Moment of the total force F , about O – O is given by : $F \times y_p$ **(3.8)**

Equating the two values given by equations (3.7) & (3.8)

$$F \times y_p = \rho\, g \sin\Theta\, I_o$$

$$y_p = \frac{\rho\, g \sin\theta\, I_o}{F} \qquad (3.9)$$

But , $\sin\Theta = \frac{h_p}{y_p}$ **,** $y_p = \frac{h_p}{\sin\theta}$ **, and** $F = \rho\, g\, A\, h_c$

And $I_o = I_G + A\, y_c^2$**, Substituting these values in eq.(3.9) , we get :**

$$\frac{h_p}{\sin\theta} = \frac{\rho\, g \sin\theta}{\rho\, g\, A h_c} (I_G + A\, y_c^2) \qquad (\times \sin\Theta)$$

But , $\sin\Theta = \frac{h_c}{y_c}$, $y_c = \frac{h_c}{\sin\theta}$

$$h_p = \frac{\sin^2\theta}{A h_c} \left(I_G + A \frac{h_c^2}{\sin^2\theta} \right)$$

$$h_p = \frac{I_G \sin^2\theta}{A\, h_c} + h_c \qquad (3.10)$$

If the $\Theta = 90^0$**, equation (3.10) becomes same as equation (3.5) (vertical plane submerged) .**

3.4 / <u>Curved Surface Submerged in Liquid</u> :

Consider a curved surface (AB) , submerged in a static liquid as shown in Fig.(3.3) . Let dA is the area of a small strip at a depth of (h) from water surface.

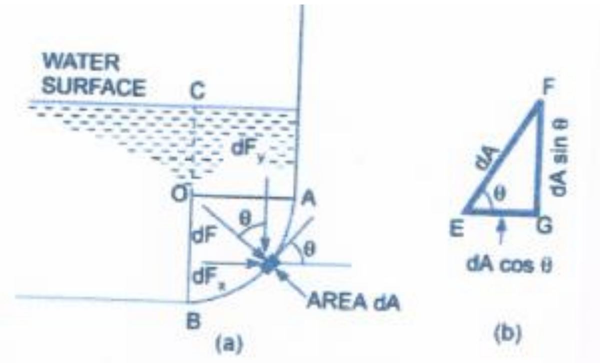


Fig.(3.3)

Then pressure (p) = ρ g h

Force (dF) = p × area = ρ g h × dA (3.11) This force dF acts normal to the surface , hence , total force on the curved surface should be:

$$F = \int \rho\, g\, h\, dA \quad (3.12)$$

By resolving the force dF in two components dF , and dF_x and dF_y in the x and y directions respectively . The total force in the x and y directions , i .e , F_x and F_y are obtained by integrating dF_x and dF_y , Then total force on the curved surface is :

$$F = \sqrt{F_x^2 + F_y^2} \qquad (3.13)$$

And inclination of resultant with horizontal is ,

$$\tan \Theta = \frac{F_y}{F_x} \qquad (3.14)$$

Resolving the force dF given by equation (3.11) in x and y directions :

$$dF_x = dF \sin\Theta = \rho\, g\, h\, dA \sin\Theta$$

$$dF_y = dF \cos\Theta = \rho\, g\, h\, dA \cos\Theta$$

Total forces in the x and y directions are :

$$F_x = \int dF_x = \rho\, g \int h\, dA \sin\Theta \qquad (3.15)$$

$$F_y = \int dF_y = \rho\, g \int h\, dA \cos\Theta \qquad (3.16)$$

Fig.(3.3) b , shows the enlarged area dA , from this figure , i.e. , Δ EFG :

$$EF = dA \;,\quad FG = dA \sin\Theta \;,\quad EG = dA \cos\Theta$$

Thus , in Eq.(3.15) , $dA \sin\Theta = FG$ = Vertical projection of the area dA .

Therefore , F_x force on the projected area on the vertical plane .

Thus , in Eq.(3.16) , $dA \cos\Theta = EG$ = Horizontal projection of the area dA.

Therefore , $\int h \, dA \cos\Theta$ is the total volume contained between the curved surface , extended up to free surface .

Hence , $\rho g \int h \, dA \cos\Theta$ is the total weight supported by the curved surface , thus , $F_y = \rho g \int h dA \cos \Theta$ = Weight of liquid supported by the curved surface up to free surface of liquid.