Ministry of Higher Education and Scientific Research

Al-Mustaqbal University College

Department of Technical Computer Engineering

Week: 5 and 6

Mathematics II

2nd Stage

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1. <u>Double integral</u>

The definite integral can be extended to functions of more than one variable. Consider, for example, a function of two variables z = f (x, y). The double integral of function f (x, y) is denoted by

$$\iint_R F(x,y)dA$$

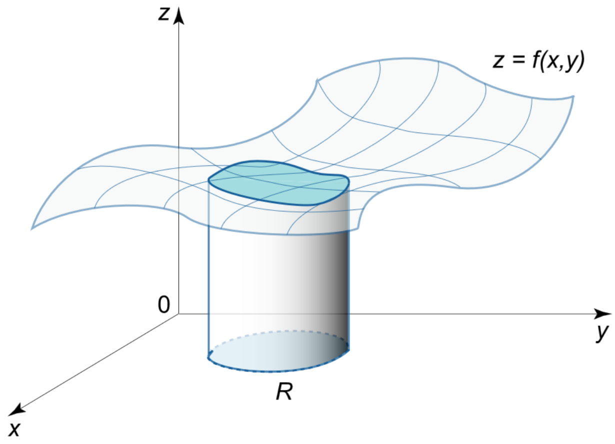


Figure 1

Where R is the region of integration in the xy-plane.

The definite integral $\int_a^b f(x)dx$ of a function of one variable f(x) ≥ 0 is the area under the curve f(x) from x=a to x=b, then the double integral is equal to the volume under the surface z=f (x, y) and above the xy-plane in the region of integration R (Figure 1).

a- <u>Properties of double integral</u>

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \quad \text{(any number } c\text{)}$$

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

(a) $$\iint_R f(x, y)\, dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \text{ on } R$$

(b) $$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA \quad \text{if} \quad f(x, y) \geq g(x, y) \text{ on } R$$

4. *Additivity:* $$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

if R is the union of two nonoverlapping regions R_1 and R_2

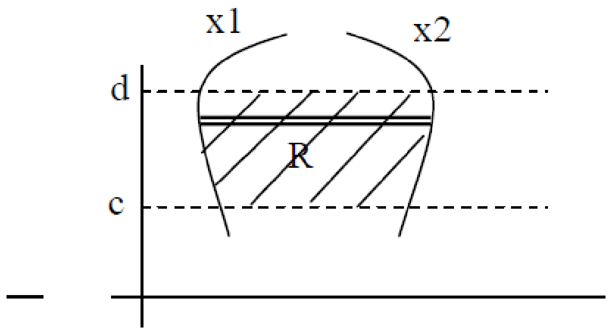
b- **Cartesian form**

Double integral of $f(x, y)$ over the region R is denoted by:

$$\iint_R F(x,y)dA = \iint_R F(x,y)\, dx\, dy = \int_c^d \int_{x1}^{x2} F(x,y)\, dx\, dy \qquad \text{Fig.2a}$$

or

$$\iint_R F(x,y)dA = \iint_R F(x,y)\, dy\, dx = \int_a^b \int_{y1}^{y2} F(x,y)\, dy\, dx \qquad \text{Fig.2b}$$



(a) (b)

Figure 2

C- **Finding Limits of Integration in cartesian form**

- **Using Vertical Cross-Sections**

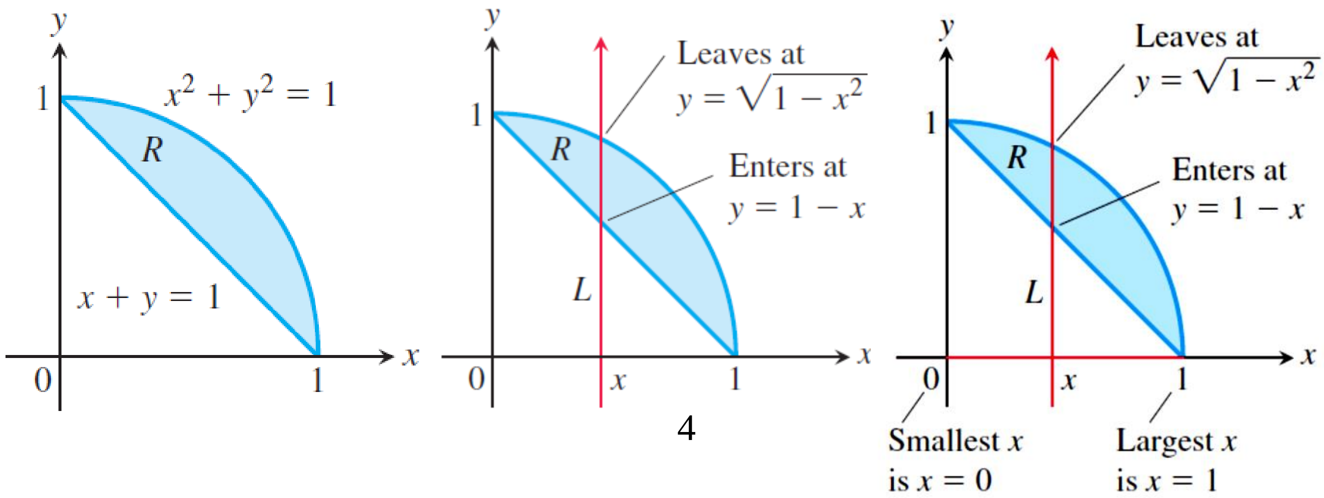
When faced with evaluating $\iint_R f(x,y)\,dA$, integrating first with respect to *y* and then with respect to *x*, do the following three steps:

1- Sketch. Sketch the region of integration and label the bounding curves. (Figure 3 a).

2- Find the y-limits of integration. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants) (Figure 3 b).

3- Find the x-limits of integration. Choose x-limits that include all the vertical lines through R. The integral shown here (see Figure 3 c) is

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$



(a) (b) (c)

Figure 3

- **Using Horizontal Cross-Sections**

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3 (see Figure 4). The integral is

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$

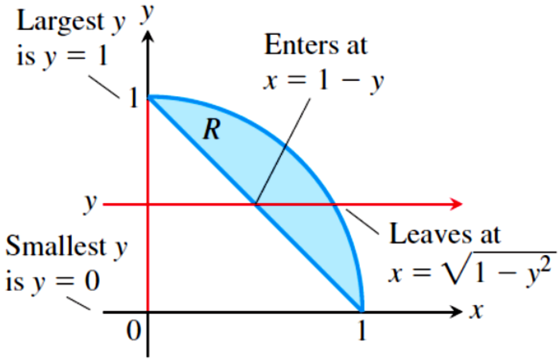


Figure 4

d- **<u>Polar form</u>**

$$\iint_R F(r,\theta)\, dA = \int_{r_1}^{r_2} \int_{\theta_1=g_1(r)}^{\theta_2=g_2(r)} F(r,\theta)\ r\, d\theta\, dr$$

or

$$\iint_R F(r,\theta)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1=g_1(\theta)}^{r_2=g_2(\theta)} F(r,\theta)\ r\, dr\, d\theta$$

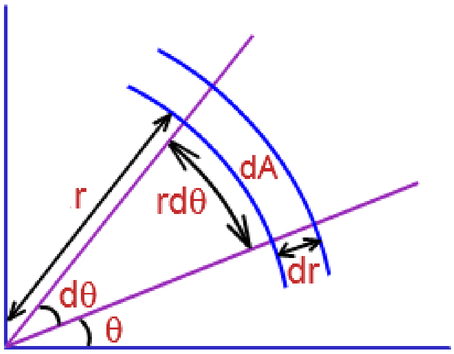


Figure 5

e- **Finding Limits of Integration in polar form**

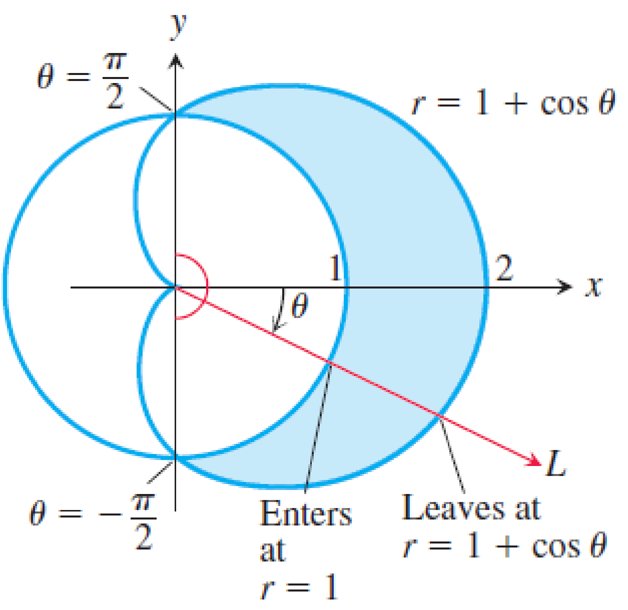
The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r,\theta)\,dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

1- *Sketch*. Sketch the region and label the bounding curves.

2- *Find the r-limits of integration*. Imagine a ray *L* from the origin cutting through *R* in the direction of increasing *r*. Mark the *r*-values where *L* enters and leaves *R*. These are the *r*-limits of integration. They usually depend on the angle u that *L* makes with the positive *x*-axis.

3- *Find the θ-limits of integration*. Find the smallest and largest θ-values that bound *R*. These are the θ-limits of integration (see figure 6). The polar iterated integral is

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\pi/4}^{\theta=\pi/2}\int_{r=\sqrt{2}\csc\theta}^{r=2} f(r,\theta)\,r\,dr\,d\theta.$$



$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\,r\,dr\,d\theta.$$

Figure 6

f- **Change of variables**

Let $x = x\,(u,v)\,, y = y\,(u,v)$ then the formula for a change of variables in double integrals from x, y to u, v is

$$\iint_R F(x,y)\,dy\,dx = \iint_{R'} F(x\,(u,v)\,, y\,(u,v))\,\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

that is, the integrand is expressed in terms of u and v, and dx, dy is replaced by du dv times
the absolute value of the Jacobian.

$$j = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

For double integral transformation from the cartesian coordinates to polar coordinates ordinates as follows:

Since $\quad x = r\cos\theta \quad , \quad y = r\sin\theta$

using the Jacobian matrix, we find that

$$j = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r((\cos\theta)^2 + (\sin\theta)^2) = r$$

Then

$$\int_{x1}^{x2}\int_{y1}^{y2} F(x,y)\,dy\,dx = \int_{r1}^{r2}\int_{\theta 1}^{\theta 2} F(r,\theta)\,r\,dr\,d\theta$$

g- **Triple integral**

If f (x, y, z) is a function defined on a closed bounded region D in space, such as the region occupied by a solid ball or a lump of clay, then the integral of f over D may be defined in the following way.

$$V = \iiint_D dV = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\, dz\, dy\, dx$$

h- **Surface area**

Let f (x, y) be a differentiable function. As we have seen, z=f(x, y) defines a surface in x y z-space. In some applications, it necessary to know the surface area of the surface above some region R in the xy-plane. See the figure.

$$S = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\; dy\, dx$$

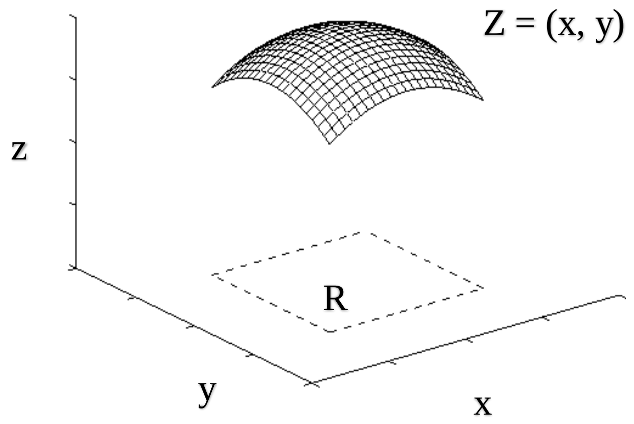


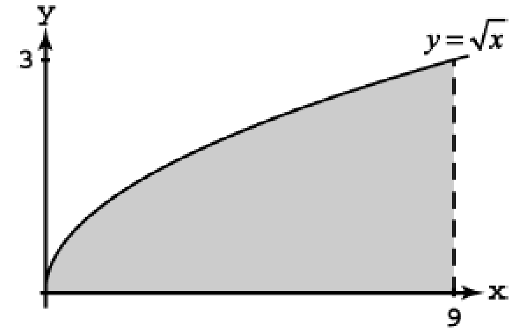
Figure 7

Examples

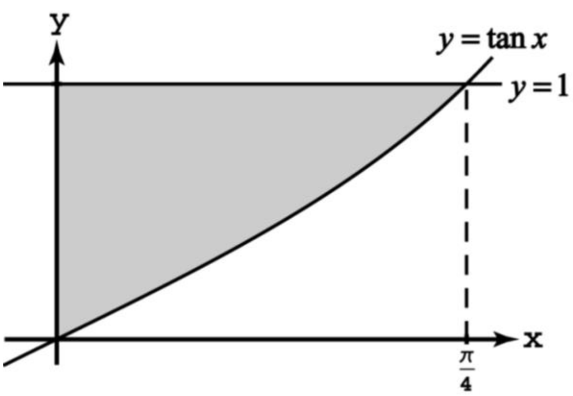
1. Double integral

a- Cartesian form

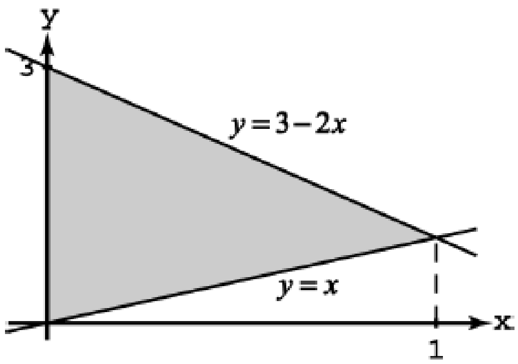
1- Find the limits of the following integral



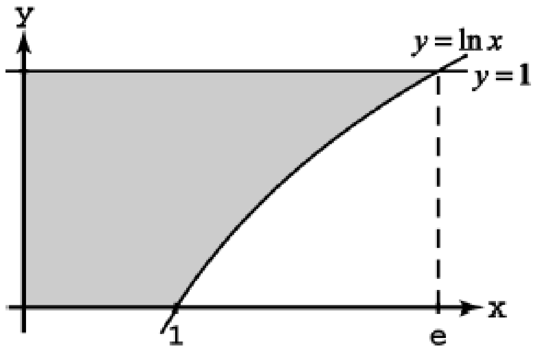
(a)



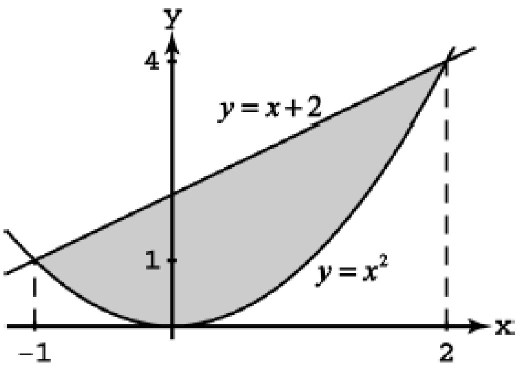
(b)



(c)



(d)



(e)

(a) $\int_0^9 \int_0^{\sqrt{x}} dy\, dx$

$\int_0^3 \int_{y^2}^9 dx\, dy$

(b) $\int_0^{\pi/4} \int_{\tan x}^1 dy\, dx$

$\int_0^1 \int_0^{\tan^{-1} y} dx\, dy$

(e) $\int_0^1 \int_x^{3-2x} dy\, dx$

$\int_0^1 \int_0^y dx\, dy + \int_1^3 \int_0^{(3-y)/2} dx\, dy$

(f) $\int_0^1 \int_0^1 dy\, dx + \int_1^e \int_{\ln x}^1 dy\, dx$

$\int_0^1 \int_0^{e^y} dx\, dy$

(g) $\int_{-1}^2 \int_{x^2}^{x+2} dy\, dx$

$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy$

2- Evaluate the following

a- $\int_0^3 \int_1^2 (1 + 8xy)\, dy\, dx$ b- $\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx\, dy$

c- $\int_0^3 \int_{-2}^0 (x^2y - 2xy)\, dy\, dx$ d- $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y)\, dx\, dy$

e- $\iint_S (\sin x + \cos y)\, dA$ bounded by the area in fig.8

f- $\iint_R xy^2\, dA$ bounded by the area in fig.9

g- $\iint_T (x - 3y)\, dA$ bounded by the area in fig.10

h- $\iint_R dA.$ bounded by the area in fig.11

i- $\iint_R dA.$ bounded by the area in fig.12

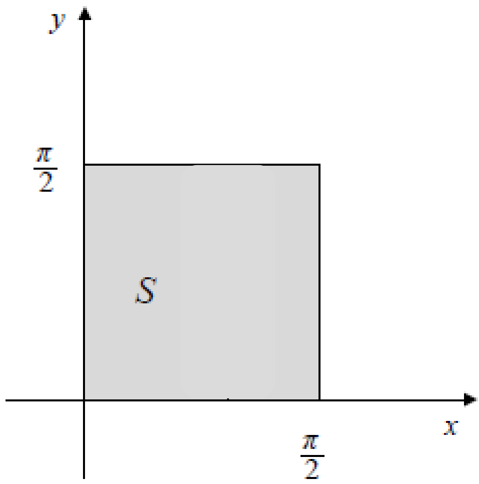


Figure 8

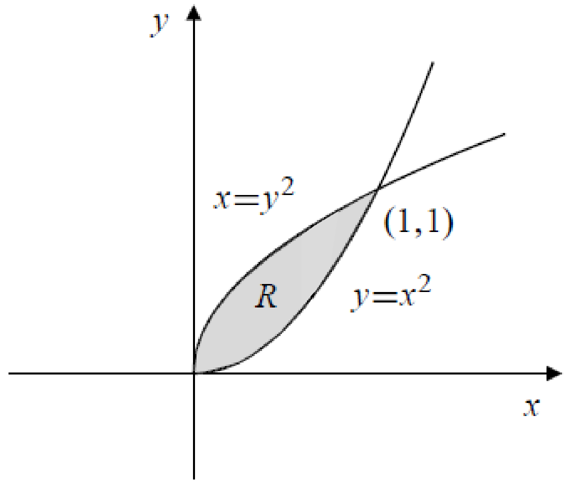


Figure 9

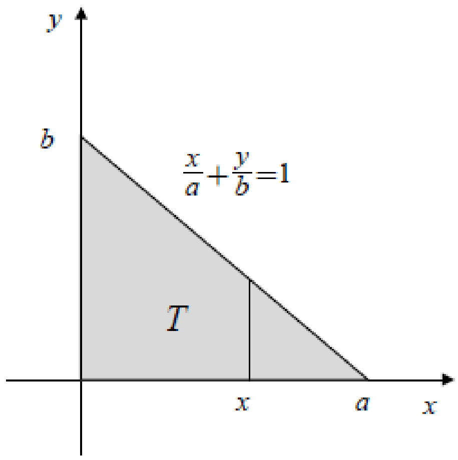
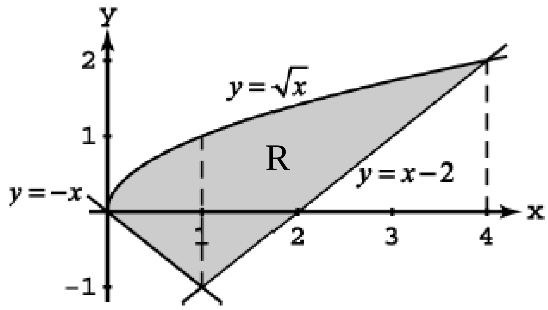


Figure 10



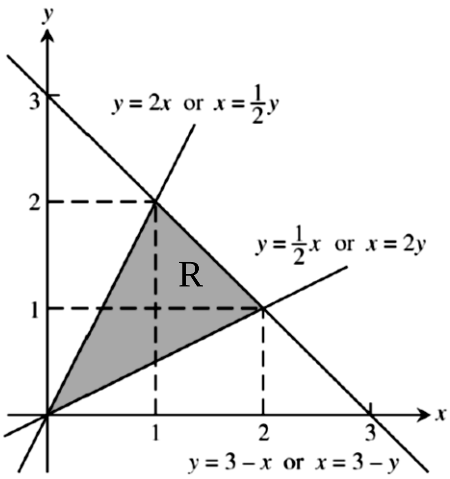


Figure 12

Solution

a- $$\int_0^3\int_1^2 (1+8xy)\,dy\,dx = \int_0^3 (y+8x\frac{y^2}{2})\Big|_1^2 dx$$
$$= \int_0^3 \{1+12x\}\,dx$$
$$= (x+12\frac{x^2}{2})\Big|_0^3$$
$$= (3+6(9)) - (0) = (3+54) = \mathbf{57}$$

b- $$\int_0^1\int_0^1\left(1-\frac{x^2+y^2}{2}\right)dx\,dy = \int_0^1\left[x-\frac{x^3}{6}-\frac{xy^2}{2}\right]_0^1 dy$$
$$= \int_0^1\left(\frac{5}{6}-\frac{y^2}{2}\right)dy = \left[\frac{5}{6}y-\frac{y^3}{6}\right]_0^1 = \frac{2}{3}$$

c- $$\int_0^3\int_{-2}^0 (x^2y-2xy)\,dy\,dx = \int_0^3\left[\frac{x^2y^2}{2}-xy^2\right]_{-2}^0 dx = \int_0^3 (4x-2x^2)\,dx$$
$$= \left[2x^2-\frac{2x^3}{3}\right]_0^3 = 0$$

d- $$\int_\pi^{2\pi}\int_0^\pi (\sin x+\cos y)\,dx\,dy = \int_\pi^{2\pi}[-\cos x + x\cos y]_0^\pi\,dy$$
$$= \int_\pi^{2\pi}(2+\pi\cos y)\,dy = [2y+\pi\sin y]_\pi^{2\pi} = 2\pi$$

e- $$\iint_S (\sin x+\cos y)\,dA$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}(\sin x+\cos y)\,dy\,dx$$
$$= \int_0^{\pi/2} dx\,\left(y\sin x+\sin y\right)\Big|_{y=0}^{y=\pi/2}$$
$$= \int_0^{\pi/2}\left(\frac{\pi}{2}\sin x+1\right)dx$$
$$= \left(-\frac{\pi}{2}\cos x + x\right)\Big|_0^{\pi/2} = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

f- $$\iint_R xy^2\,dA = \int_0^1 x\,dx \int_{x^2}^{\sqrt{x}} y^2\,dy$$
$$= \int_0^1 x\,dx \left(\frac{1}{3}y^3\right)\Big|_{y=x^2}^{y=\sqrt{x}}$$
$$= \frac{1}{3}\int_0^1 \left(x^{5/2} - x^7\right)dx$$
$$= \frac{1}{3}\left(\frac{2}{7}x^{7/2} - \frac{x^8}{8}\right)\Big|_0^1$$
$$= \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right) = \frac{3}{56}.$$

g- $$\iint_T (x-3y)\,dA = \int_0^a dx \int_0^{b(1-(x/a))} (x-3y)\,dy$$
$$= \int_0^a dx \left(xy - \frac{3}{2}y^2\right)\Big|_{y=0}^{y=b(1-(x/a))}$$
$$= \int_0^a \left[b\left(x - \frac{x^2}{a}\right) - \frac{3}{2}b^2\left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right)\right]dx$$
$$= \left(b\frac{x^2}{2} - \frac{b}{a}\frac{x^3}{3} - \frac{3}{2}b^2x + \frac{3}{2}\frac{b^2x^2}{a} - \frac{1}{2}\frac{b^2x^3}{a^2}\right)\Big|_0^a$$
$$= \frac{a^2b}{6} - \frac{ab^2}{2}.$$

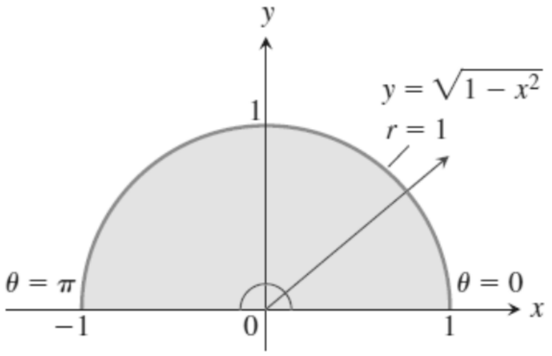
h- $$\int_0^1\int_{x/2}^{2x} 1\,dy\,dx + \int_1^2\int_{x/2}^{3-x} 1\,dy\,dx$$
$$= \int_0^1 [y]_{x/2}^{2x}dx + \int_1^2 [y]_{x/2}^{3-x}dx$$
$$= \int_0^1 \left(\tfrac{3}{2}x\right)dx + \int_1^2 \left(3 - \tfrac{3}{2}x\right)dx$$
$$= \left[\tfrac{3}{4}x^2\right]_0^1 + \left[3x - \tfrac{3}{4}x^2\right]_1^2 = \tfrac{3}{2}$$

i- $$\int_0^1\int_{-x}^{\sqrt{x}} 1\,dy\,dx + \int_1^4\int_{x-2}^{\sqrt{x}} 1\,dy\,dx$$
$$= \int_0^1 [y]_{-x}^{\sqrt{x}}dx + \int_1^4 [y]_{x-2}^{\sqrt{x}}dx$$
$$= \int_0^1 (\sqrt{x} + x)dx + \int_1^4 (\sqrt{x} - x + 2)dx$$

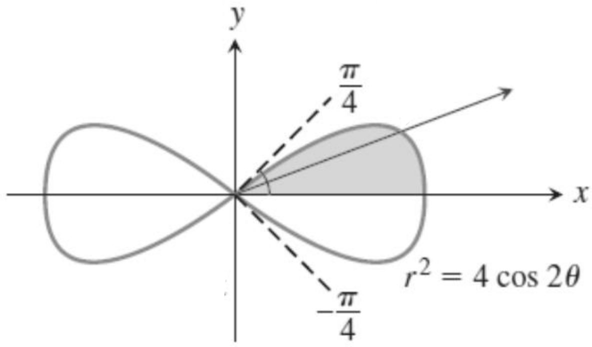
$$= \left[\tfrac{2}{3}x^{3/2} + \tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{2}x^2 + 2x\right]_1^4 = \tfrac{13}{3}$$

b- <u>Poalr form</u>

1- Find the limits of the following integral



(a)



(b)

a- $\displaystyle\int_0^{\pi}\int_0^1 r\,dr\,d\theta$

b- $\displaystyle\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$

2- Evaluate the following

a- $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r^2\cos\theta\,dr\,d\theta$

b- $\displaystyle\int_0^{\pi/4}\int_0^{2\sec\theta} r^2\sin\theta\,dr\,d\theta$

c- $\displaystyle\int_{\pi/6}^{\pi/4}\int_{\csc\theta}^{\sqrt{3}\sec\theta} r\,dr\,d\theta$

d- $\displaystyle\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4}\,r\,dr\,d\theta$

e- $\displaystyle\iint_S x\,dA$ bounded by area shown in fig.13

f- $\displaystyle\iint_C y\,dA$ bounded by area shown in fig.14

g- $\displaystyle\iint_A \boldsymbol{dA}$ bounded by area shown in fig.15

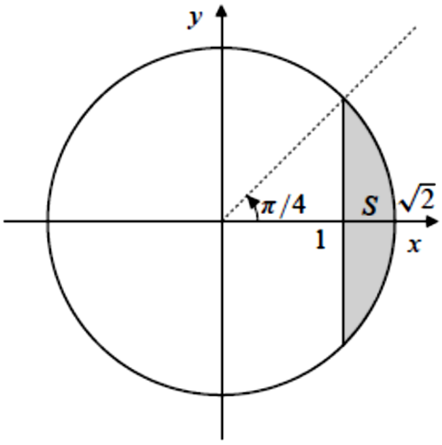
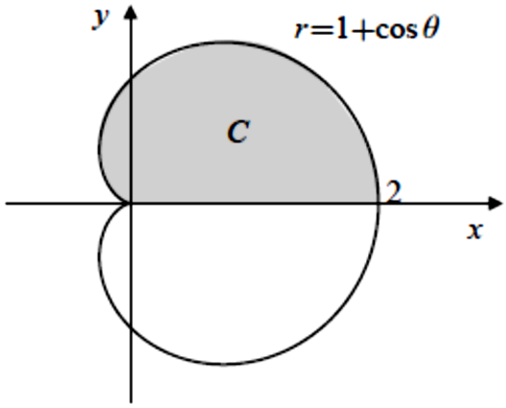


Figure 13



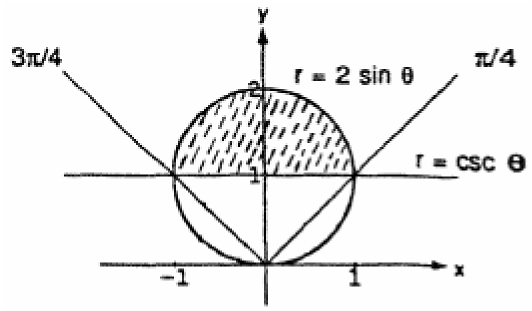


Figure 15

Solution

a- $$\int_{\pi/4}^{\pi/2}\int_{0}^{6\csc\theta} r^2\cos\theta\, dr\, d\theta = 72\int_{\pi/4}^{\pi/2}\cot\theta\csc^2\theta\, d\theta$$

$$= -36\,[\cot^2\theta]_{\pi/4}^{\pi/2} = 36$$

b- $$\int_{0}^{\pi/4}\int_{0}^{2\sec\theta} r^2\sin\theta\, dr\, d\theta = \frac{8}{3}\int_{0}^{\pi/4}\tan\theta\sec^2\theta\, d\theta = \frac{4}{3}$$

c- $$\int_{\pi/6}^{\pi/4}\int_{\csc\theta}^{\sqrt{3}\sec\theta} r\, dr\, d\theta = \int_{\pi/6}^{\pi/4}\left(\frac{3}{2}\sec^2\theta - \frac{1}{2}\csc^2\theta\right) d\theta$$

$$= \left[\frac{3}{2}\tan\theta + \frac{1}{2}\cot\theta\right]_{\pi/6}^{\pi/4} = 2-\sqrt{3}$$

$$\int_{0}^{\pi/4}\int_{\sec\theta}^{2\cos\theta}\frac{1}{r^4}\, r\, dr\, d\theta = \int_{0}^{\pi/4}\left[-\frac{1}{2r^2}\right]_{\sec\theta}^{2\cos\theta} d\theta = \int_{0}^{\pi/4}\left(\frac{1}{2}\cos^2\theta - \frac{1}{8}\sec^2\theta\right) d\theta$$

$$= \left[\frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{1}{8}\tan\theta\right]_{0}^{\pi/4} = \frac{\pi}{16}$$

d-

e- $$\iint_S x\,dA = 2\int_0^{\pi/4}\int_{\sec\theta}^{\sqrt{2}} r\cos\theta\, r\,dr\,d\theta$$
$$= \frac{2}{3}\int_0^{\pi/4} \cos\theta\left(2\sqrt{2} - \sec^3\theta\right)d\theta$$
$$= \frac{4\sqrt{2}}{3}\sin\theta\Big|_0^{\pi/4} - \frac{2}{3}\tan\theta\Big|_0^{\pi/4}$$
$$= \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

f- $$\iint_C y\,dA = \int_0^{\pi}\int_0^{1+\cos\theta} r\sin\theta r\,dr\,d\theta$$
$$= \frac{1}{3}\int_0^{\pi}\sin\theta(1+\cos\theta)^3\,d\theta \quad \text{Let } u = 1+\cos\theta$$
$$du = -\sin\theta\,d\theta$$
$$= \frac{1}{3}\int_0^{2} u^3\,du = \frac{u^4}{12}\Big|_0^2 = \frac{4}{3}$$

g- $$\int_{\pi/4}^{3\pi/4}\int_{\csc\theta}^{2\sin\theta} \text{r dr d}\theta = \tfrac{1}{2}\int_{\pi/4}^{3\pi/4}(4\sin^2\theta - \csc^2\theta)\,\text{d}\theta$$
$$= \tfrac{1}{2}\left[2\theta - \sin 2\theta + \cot\theta\right]_{\pi/4}^{3\pi/4} = \tfrac{\pi}{2}$$

c- <u>Change of variables</u>

Evaluate the following integrals in polar form

a- $$\int_0^2\int_0^{\sqrt{4-y^2}}(x^2+y^2)\,dx\,dy$$

b- $$\int_0^2\int_0^{x} y\,dy\,dx$$

$$\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dx\,dy$$

c-

d- $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1 + x^2 + y^2)^2}\, dy\, dx$

e- $\int_{0}^{1}\int_{x}^{\sqrt{2-x^2}} (x + 2y)\, dy\, dx$

f- $\iint_S (x + y)\, dA$ where s is the area bounded in fig.16

g- $\iint_T (x^2 + y^2)\, dA$ where T is the area bounded in fig.17

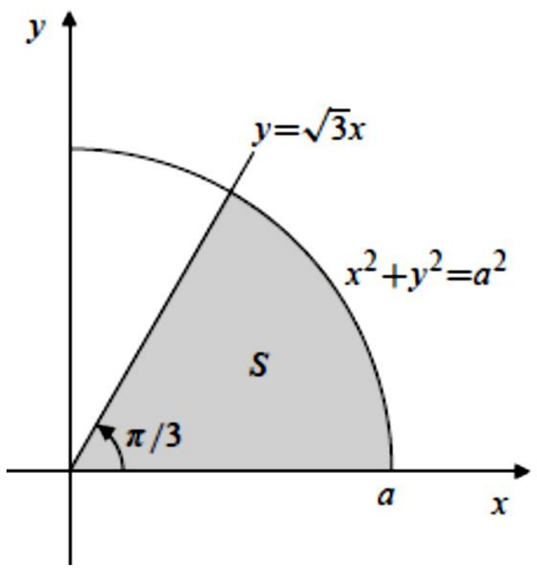


Figure 17

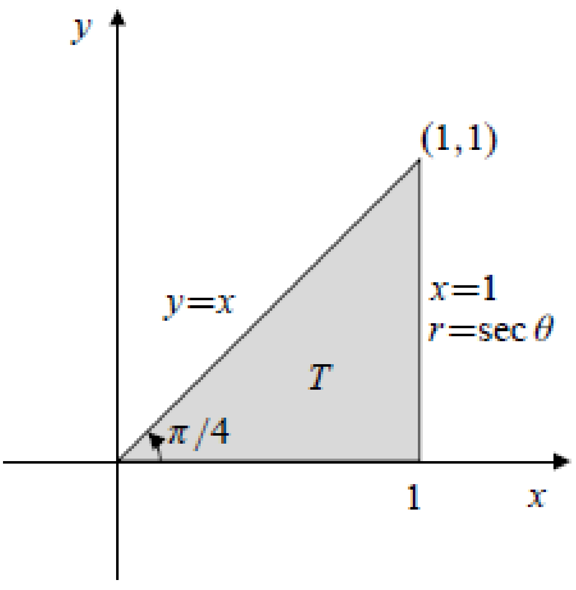
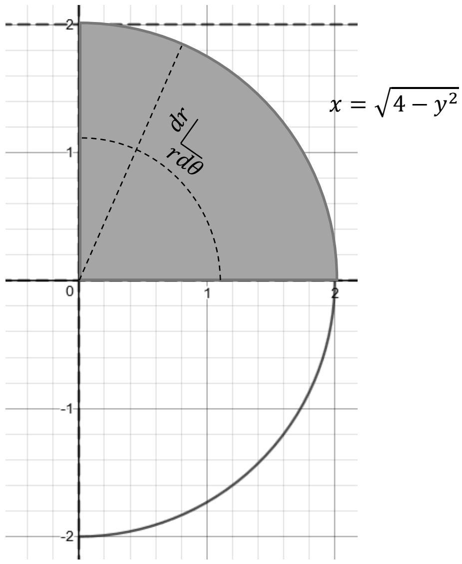


Figure 16

Solution

a- $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)\, dx\, dy$



$x_2 = \sqrt{4-y^2}$, $x_1 = 0$

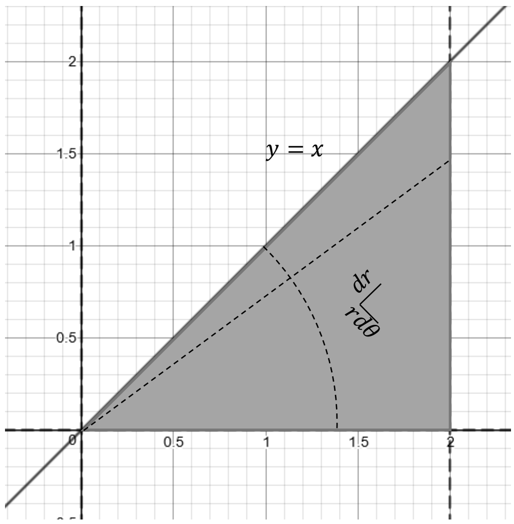
$y_2 = 2$, $y_1 = 0$

$f(x,y) = x^2 + y^2 = r^2$, $dxdy = rdrd\theta$

$$\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2)\, dx\, dy = \int_{\theta_1=0}^{\theta_2=\frac{\pi}{2}} \int_{r_1=0}^{r_2=2} r^2 \; rdrd\theta$$

$$= \int_{\theta_1=0}^{\theta_2=\frac{\pi}{2}} \int_{r_1=0}^{r_2=2} r^3 \; drd\theta = \int_0^{\pi/2} \frac{r^4}{4}\bigg|_0^2 \; drd\theta = 4\int_0^{\pi/2} d\theta = 2\pi$$

b- $\int_0^2 \int_0^x y\, dy\, dx$



$y_2 = x$, $y_1 = 0$

$x_2 = 2$, $x_1 = 0$

$f(x,y) = y = r\sin\theta$, $dxdy = rdrd\theta$

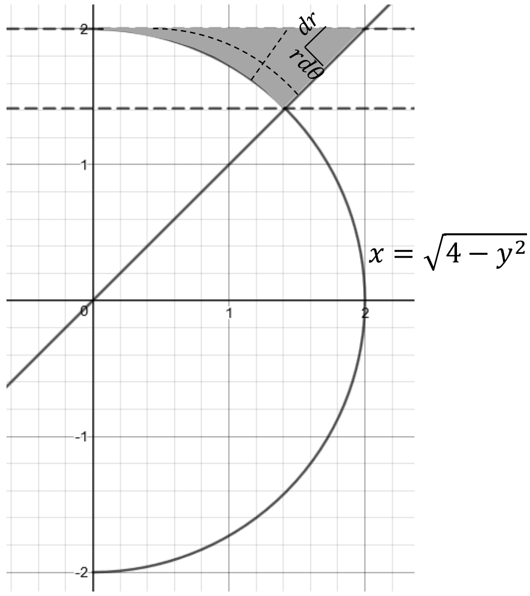
$$\int_0^2 \int_0^x y\, dy\, dx = \int_0^{\pi/4} \int_0^{2\sec\theta} r^2 \sin\theta\, dr\, d\theta$$

$$= \frac{8}{3}\int_0^{\pi/4} \tan\theta \sec^2\theta\, d\theta = \frac{4}{3}$$

c- $\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dx\, dy$

$x_2 = y \quad , \quad x_1 = \sqrt{4-y^2}$

$y_2 = 2 \quad , \quad y_1 = \sqrt{2}$



$f(x,y) = 1 \quad , \quad dxdy = rdrd\theta$

$$\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} \mathrm{dy\,dx} = \int_{\pi/4}^{\pi/2}\int_{2}^{2\csc\theta} \mathrm{r\,dr\,d}\theta$$

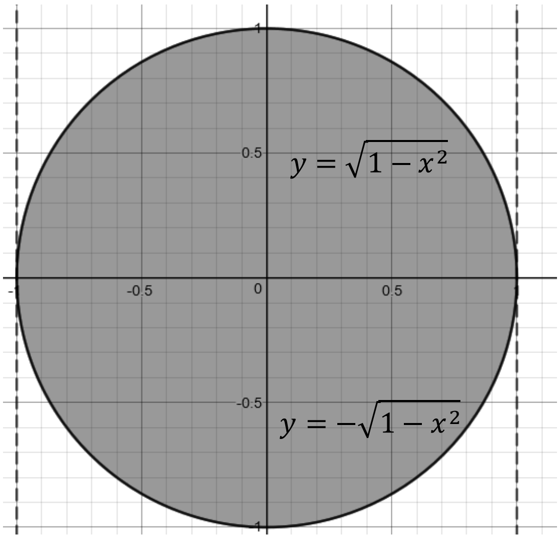
$$= \int_{\pi/6}^{\pi/4}(2\csc^2\theta - 2)\,\mathrm{d}\theta = \left[-2\cot\theta - \tfrac{1}{2}\theta\right]_{\pi/4}^{\pi/2}$$

$$= 2 - \frac{\pi}{2}$$

d- $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1\,+\,x^2\,+\,y^2)^2}\, dy\, dx$

$y_2 = \sqrt{1-x^2} \quad , \quad y_1 = -\sqrt{1-x^2}$

$x_2 = 1 \quad , \quad x_1 = -1$

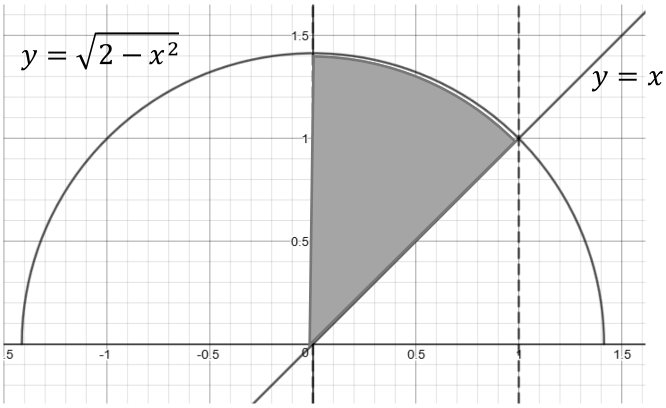


$\frac{2}{(1+x^2+y^2)} = \frac{2}{(1+r^2)} \quad , \quad dxdy = rdrd\theta$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,\mathrm{dy\,dx} =$$

$$= 4\int_{0}^{\pi/2}\int_{0}^{1} \frac{2r}{(1+r^2)^2}\,\mathrm{dr\,d}\theta = 4\int_{0}^{\pi/2}\left[-\frac{1}{1+r^2}\right]_{0}^{1}\mathrm{d}\theta = 2\int_{0}^{\pi/2}\mathrm{d}\theta = \pi$$

e- $\int_0^1 \int_x^{\sqrt{2-x^2}} (x + 2y)\,dy\,dx$



$x_2 = 1 \quad , \quad x_1 = 0$

$y_2 = \sqrt{2 - x^2} \quad , \quad y_1 = x$

$(x + 2y) = r(\cos\theta + 2\sin\theta) \quad ,$

$dxdy = rdrd\theta$

$$\int_0^1 \int_x^{\sqrt{2-x^2}} (x + 2y)\,dy\,dx = \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} (r\cos\theta + 2r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^3}{3}\cos\theta + \frac{2r^3}{3}\sin\theta\right]_0^{\sqrt{2}} d\theta = \int_{\pi/4}^{\pi/2} \left(\frac{2\sqrt{2}}{3}\cos\theta + \frac{4\sqrt{2}}{3}\sin\theta\right) d\theta$$

$$= \left[\frac{2\sqrt{2}}{3}\sin\theta - \frac{4\sqrt{2}}{3}\cos\theta\right]_{\pi/4}^{\pi/2} = \frac{2(1+\sqrt{2})}{3}$$

f- $\iint_S (x + y)\,dA = \int_0^{\pi/3} d\theta \int_0^a (r\cos\theta + r\sin\theta) r\,dr$

$$= \int_0^{\pi/3} (\cos\theta + \sin\theta)\,d\theta \int_0^a r^2\,dr$$

$$= \frac{a^3}{3}(\sin\theta - \cos\theta)\Big|_0^{\pi/3}$$

$$= \left[\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) - (-1)\right]\frac{a^3}{3} = \frac{(\sqrt{3}+1)a^3}{6}$$

h- $\iint_T (x^2 + y^2)\,dA = \int_0^{\pi/4} d\theta \int_0^{\sec\theta} r^3\,dr$

$$= \frac{1}{4}\int_0^{\pi/4} \sec^4\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/4} (1 + \tan^2\theta)\sec^2\theta\,d\theta \quad \text{Let } u = \tan\theta$$

$$du = \sec^2\theta\,d\theta$$

$$= \frac{1}{4}\int_0^1 (1+u^2)\,du$$

$$= \frac{1}{4}\left(u + \frac{u^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

d- <u>Triple integral</u>

Evaluate the following integrals:

a- $\int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\,dz\,dy\,dx$

b- $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$

c- $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy$

d- $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\,dx\,dy\,dz$

e- $\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx$

Solution

a- $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx = \int_0^1\int_0^1 \left(x^2+y^2+\frac{1}{3}\right)dy\,dx$

$$= \int_0^1\left(x^2+\frac{2}{3}\right)dx = 1$$

b- $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy = \int_0^{\sqrt{2}}\int_0^{3y}(8-2x^2-4y^2)\,dx\,dy$

$$= \int_0^{\sqrt{2}}\left[8x - \frac{2}{3}x^3 - 4xy^2\right]_0^{3y} dy = \int_0^{\sqrt{2}}(24y - 18y^3 - 12y^3)\,dy$$

$$= \left[12y^2 - \frac{15}{2}y^4\right]_0^{\sqrt{2}} = 24 - 30 = -6$$

c- $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy = \int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(2x+y)\,dx\,dy$

$$= \int_0^2 [x^2 + xy]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy = \int_0^2 (4 - y^2)^{1/2} (2y)\, dy$$

$$= \left[-\tfrac{2}{3}(4 - y^2)^{3/2}\right]_0 = \tfrac{2}{3}(4)^{3/2} = \tfrac{16}{3}$$

d- $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \, dx\, dy\, dz = \int_0^{\pi/6} \int_0^1 5y \sin z \, dy\, dz$

$$= \tfrac{5}{2} \int_0^{\pi/6} \sin z \, dz = \frac{5(2-\sqrt{3})}{4}$$

e- $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \, dz\, dy\, dx = \int_0^1 \int_0^{1-x^2} x(1 - x^2 - y)\, dy\, dx$

$$= \int_0^1 x\left[(1-x^2)^2 - \tfrac{1}{2}(1-x^2)\right] dx = \int_0^1 \tfrac{1}{2} x (1-x^2)^2\, dx$$

$$= \left[-\tfrac{1}{12}(1-x^2)^3\right]_0^1 = \tfrac{1}{12}$$

e- <u>Surface area</u>

Find the area of the following surfaces:

a- $Z = f(x,y) = 6 - 3x - 2y$ lies in the region shown in fig. 18

b- $Z = x^2 + y^2$ lies in the region shown in fig. 19

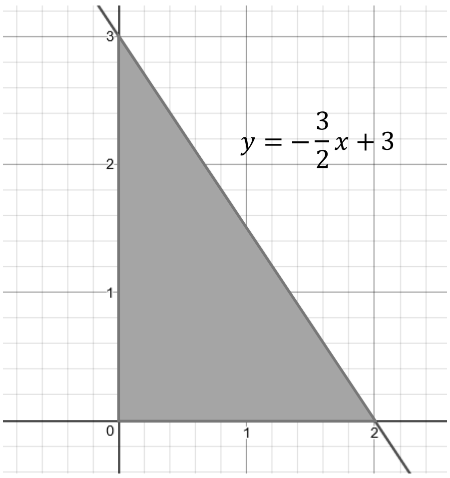


Figure 18

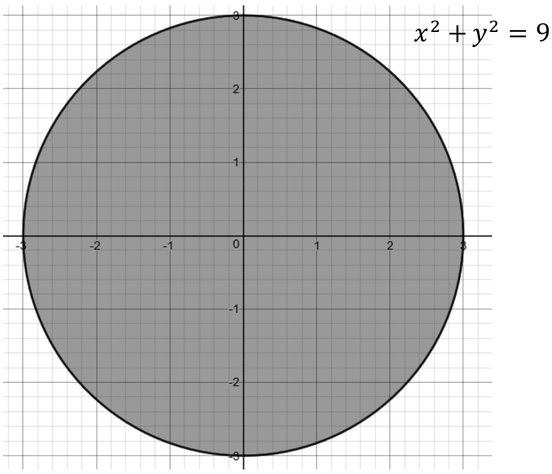


Figure 19

Solution

a- $Z = 6 - 3x - 2y$

$$\frac{\partial f}{\partial x} = -3 \quad , \quad \frac{\partial f}{\partial y} = -2 \quad , \quad 0 \le x \le 2 \quad , \quad 0 \le y \le -\frac{3}{2}x + 3$$

$$S = \iint\limits_{R} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dA$$

$$= \int_0^2 \int_0^{-\frac{3}{2}x+3} \sqrt{(-3)^2 + (-2)^2 + 1}\ dy\, dx = \sqrt{14} \int_0^2 \left(-\frac{3}{2}\ x + 3\right) dx$$

$$S = \sqrt{14}\ \left(-\frac{3}{4}x^2 + 3x\right)\Big|_0^2 = 3\sqrt{14}$$

b- $Z = x^2 + y^2$

$$\frac{\partial f}{\partial x} = 2x \quad , \quad \frac{\partial f}{\partial y} = 2y$$

$$S = \iint\limits_{R} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dA$$

$$= \iint\limits_{R} \sqrt{1 + 4x^2 + 4y^2}\ dA \qquad\qquad r^2 = x^2 + y^2$$

$$= \int_0^{2\pi} \int_0^3 r\ \sqrt{1 + 4r^2}\ dr d\theta = \frac{2}{24} \int_0^{2\pi} [1 + 4r^2]^{\frac{3}{2}} \Big|_0^3 d\theta = 18 \int_0^{2\pi} d\theta = 36\pi$$