Limit of a Function:.

If ***f*** is a function, then we say:

A is the limit of ***f*** **(*x*)** as ***x*** approaches ***a*** if the value of *f* (*x*) gets arbitrarily close to ***A*** as ***x*** approaches ***a***. This is written in mathematical notation as:

$$\lim_{x \to a} f(x) = A$$

For example:

$$\lim_{x \to 3} x^2 = 9$$

$$\lim_{x \to 0} x = 0$$

$$\lim_{x \to 2} 4 = 4$$

The limit law:.

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x \to c}(f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $\lim_{x \to c}(f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$$

Example:.

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$ Sum and Difference Rules

$= c^3 + 4c^2 - 3$ Product and Multiple Rules

(b) $\lim_{x\to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)}$ Quotient Rule

$= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5}$ Sum and Difference Rules

$= \frac{c^4 + c^2 - 1}{c^2 + 5}$ Power or Product Rule

(c) $\lim_{x\to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x\to -2}(4x^2 - 3)}$ Power Rule with $r/s = 1/2$

$= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3}$ Difference Rule

$= \sqrt{4(-2)^2 - 3}$ Product and Multiple Rules

$= \sqrt{16 - 3}$

$= \sqrt{13}$

Indeterminate quantities :.

$\frac{0}{0}$,	$\frac{\infty}{\infty}$,	$0*\infty$,	$\infty - \infty$,	∞^0 ,	0^0

Example:. $\lim_{x\to -2} \frac{-2x-4}{x^3+2x^2}$

Solution: $\frac{-2*-2-4}{(-2)^3+2(-2)^2} = \frac{0}{0} = \infty$, then the solution

$\lim_{x\to -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x\to -2} \frac{-2}{x^2} = \frac{-2}{(-2)^2} = \frac{-2}{4} = \frac{-1}{2}$

Example:. $\lim_{x\to 2} \frac{x^2-4}{x-2}$ $\frac{0}{0} = \infty$

Solution: $\lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} (x+2) = 2 + 2 = 4$

Example:. $\lim_{x\to 0} \frac{x}{1-\sqrt{1-x}}$ $\frac{0}{0} = \infty$

Solution: $\lim_{x\to 0} \frac{x\,(1+\sqrt{1-x}\,)}{(1-\sqrt{1-x})(\,1+\sqrt{1-x}\,)} = \lim_{x\to 0} \frac{x\,(1+\sqrt{1-x}\,)}{1-(\,1-x\,)}$

$= \lim_{x\to 0} \frac{x\,(1+\sqrt{1-x}\,)}{1-\,1+x} = \lim_{x\to 0} \frac{x\,(1+\sqrt{1-x}\,)}{x} = \lim_{x\to 0} (1+\sqrt{1-x}\,) = 2$

Example:. $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$ $\frac{0}{0} = \infty$

Solution: $\lim_{x\to 0} \frac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2\,(\sqrt{x^2+100}+10)}$

$= \lim_{x\to 0} \frac{x^2+100-100}{x^2\,(\sqrt{x^2+100}+10)} = \lim_{x\to 0} \frac{x^2}{x^2\,(\sqrt{x^2+100}+10)}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{\sqrt{0+100}+10} = \frac{1}{10+10} = \frac{1}{20}$

Limits of trigonometric functions:.

If a is constant.

$\lim_{x\to 0} \sin x = 0$	$\lim_{x\to 0} \cos x = 1$	$\lim_{x\to 0} \tan x = 0$
$\lim_{x\to 0} \frac{\sin x}{x} = 1$	$\lim_{x\to 0} \frac{\sin ax}{ax} = 1$	$\lim_{x\to 0} \frac{\tan x}{x} = 1$
$\lim_{x\to 0} \frac{\tan ax}{ax} = 1$	$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$	

Example:. Find

1. $\lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{1}{5} \lim_{x\to 0} \frac{\sin 2x}{x} = \frac{1}{5} \lim_{x\to 0} \frac{\sin 2x}{x} * \frac{2}{2}$

$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x} = \frac{2}{5} * 1 = \frac{2}{5}$

2. $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \lim_{x\to 0} \frac{\sin 2x \frac{2x}{2x}}{\sin 3x \frac{3x}{3x}} = \lim_{x\to 0} \frac{2x \frac{\sin 2x}{2x}}{3x \frac{\sin 3x}{3x}}$

$= \frac{2}{3} \lim_{x\to 0} \frac{\frac{\sin 2x}{2x}}{\frac{\sin 3x}{3x}} = \frac{2}{3} \frac{\lim_{x\to 0} \frac{\sin 2x}{2x}}{\lim_{x\to 0} \frac{\sin 3x}{3x}} = \frac{2}{3} * \frac{1}{1} = \frac{2}{3}$

3. $\lim\limits_{\theta\to 0} \dfrac{\theta}{\sin\theta} = \lim\limits_{\theta\to 0} \dfrac{1}{\frac{\sin\theta}{\theta}} = \dfrac{1}{\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}} = \dfrac{1}{1} = 1$

4. $\lim\limits_{\theta\to 0} \dfrac{\sin^2\theta}{\theta^2} = \left(\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}\right)^2 = (1)^2 = 1$

or $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} * \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1 * 1 = 1$

5. $\lim\limits_{h\to 0} \dfrac{\cos h - 1}{h} = \lim\limits_{h\to 0} \dfrac{-2\sin^2(\frac{h}{2})}{h}$

because [$\sin^2 h = \dfrac{1-\cos 2h}{2}$, $\sin^2(\frac{h}{2}) = \dfrac{1-\cos h}{2}$]

$= -\lim\limits_{h\to 0} \dfrac{\sin^2(\frac{h}{2})}{\frac{h}{2}} = -\lim\limits_{h\to 0} \dfrac{\sin(\frac{h}{2})}{\frac{h}{2}} * \lim\limits_{h\to 0} \sin(\frac{h}{2}) = -1 * 0 = 0$

6. $\lim\limits_{t\to 0} \dfrac{\tan t \ \sec 2t}{3t} = \lim\limits_{t\to 0} \dfrac{\frac{\sin t}{\cos t} * \frac{1}{\cos 2t}}{3t}$

$= \dfrac{1}{3} \lim\limits_{t\to 0}\left(\dfrac{\sin t}{t} \ \dfrac{1}{\cos t} \ \dfrac{1}{\cos 2t}\right) = \dfrac{1}{3} \lim\limits_{t\to 0} \dfrac{\sin t}{t} \ . \lim\limits_{t\to 0} \dfrac{1}{\cos t} \ . \lim\limits_{t\to 0} \dfrac{1}{\cos 2t}$

$= \dfrac{1}{3} * 1 * 1 * 1 = \dfrac{1}{3}$

7. $\lim\limits_{x\to 0} \dfrac{x\sin x}{2-2\cos x} = \lim\limits_{x\to 0} \dfrac{(x\sin x)(2+2\cos x)}{(2-2\cos x)(2+2\cos x)}$

$= \lim\limits_{x\to 0} \dfrac{(x\sin x)(2+2\cos x)}{4-4\cos^2 x} = \lim\limits_{x\to 0} \dfrac{(x\sin x)(2+2\cos x)}{4(1-\cos^2 x)}$

$= \lim\limits_{x\to 0} \dfrac{(x\sin x)(2+2\cos x)}{4\sin^2 x} \quad \lim\limits_{x\to 0} \dfrac{x(2+2\cos x)}{4\sin x}$

$= \lim\limits_{x\to 0} \dfrac{2+2\cos x}{4\frac{\sin x}{x}} = \dfrac{2+2}{4*1} = \dfrac{4}{4} = 1$

8. $\lim_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}}$ [let $y = x - \frac{\pi}{4}$, $x \to \frac{\pi}{4}$, $y \to 0$]

$$= \lim_{y \to 0} \frac{\tan(y + \frac{\pi}{4}) - 1}{y} = \lim_{y \to 0} [\frac{1}{y} (\tan(y + \frac{\pi}{4}) - 1)]$$

$$[\tan(y + \frac{\pi}{4}) = \frac{\tan y + \tan\frac{\pi}{4}}{1 - \tan y * \tan\frac{\pi}{4}} = \frac{\tan y + 1}{1 - \tan y}]$$

$$= \lim_{y \to 0} [\frac{1}{y} * (\frac{\tan y + 1}{1 - \tan y} - 1)] = \lim_{y \to 0} [\frac{1}{y} (\frac{\tan y + 1 - (1 - \tan y)}{1 - \tan y})]$$

$$= \lim_{y \to 0} [\frac{1}{y} (\frac{\tan y + 1 - 1 + \tan y}{1 - \tan y})] = \lim_{y \to 0} [\frac{1}{y} (\frac{\tan y + \tan y}{1 - \tan y})]$$

$$= \lim_{y \to 0} [\frac{1}{y} (\frac{2\tan y}{1 - \tan y})] = 2 \lim_{y \to 0} [\frac{\tan y}{y} * \frac{1}{1 - \tan y}]$$

$$= 2 \lim_{y \to 0} \frac{\tan y}{y} * \lim_{y \to 0} \frac{1}{1 - \tan y} = 2*1*1=2$$

<u>Infinite limits [as x → ± ∞] :.</u>

The basic facts to be verified by applying the formal definition are.

$$\lim_{x \to \pm\infty} k = k \quad \text{and} \quad \lim_{x \to \pm\infty} \frac{1}{x} = 0$$

Example :. $\lim_{x \to \infty} (5 + \frac{1}{x}) = \lim_{x \to \infty} 5 + \lim_{x \to \infty} \frac{1}{x} = 5 + 0 = 5$

Example :. $\lim_{x \to -\infty} \frac{\pi\sqrt{3}}{x^2} = \lim_{x \to -\infty} \pi\sqrt{3} \, \frac{1}{x} * \frac{1}{x} = \lim_{x \to -\infty} \pi\sqrt{3} * \lim_{x \to -\infty} \frac{1}{x} * \lim_{x \to -\infty} \frac{1}{x} = \pi\sqrt{3} * 0 * 0 = 0$

Example :. $\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{\frac{5x^2}{x^2}+\frac{8x}{x^2}-\frac{3}{x^2}}{\frac{3x^2}{x^2}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}}$

$= \frac{5+0+0}{3+0} = \frac{5}{3}$

Example :. $\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{\frac{11x}{x^3}+\frac{2}{x^3}}{\frac{2x^3}{x^3}-\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}$

$= \frac{0+0}{2-0} = \frac{0}{2} = 0$

Example :. $\lim_{x\to\infty} \sin(\frac{1}{x}) = \sin(\frac{1}{x}) = \sin(0) = 0$

Example :. $\lim_{x\to\infty} x\sin(\frac{1}{x})$ $[\text{ let } t = \frac{1}{x} \text{ , } x \to\infty \text{ , } t\to 0]$

$= \lim_{t\to 0} \frac{1}{t} \sin(0) = \lim_{t\to 0} \frac{\sin t}{t} = 1$

Example :. $\lim_{x\to\infty} (x - \sqrt{x^2 + 16}) = \lim_{x\to\infty} (x - \sqrt{x^2 + 16}) * \frac{(x+ \sqrt{x^2+ 16})}{(x+ \sqrt{x^2+ 16})}$

$= \lim_{x\to\infty} \frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} = \lim_{x\to\infty} \frac{x^2-x^2-16}{x+\sqrt{x^2+16}} = \lim_{x\to\infty} \frac{-16}{x+\sqrt{x^2+16}}$

$= \lim_{x\to\infty} \frac{\frac{-16}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{16}{x^2}}} = \lim_{x\to\infty} \frac{\frac{-16}{x}}{1+\sqrt{1+\frac{16}{x^2}}} = \frac{0}{1+\sqrt{1+0}} = \frac{0}{2} = 0$

H.W: Find .

1. $\lim_{x\to 2} \frac{x+3}{x+6}$ $\left(\frac{5}{8}\right)$	2. $\lim_{x\to 2/3} 3x(2x-1)$ $\left(\frac{2}{3}\right)$
3. $\lim_{x\to 5} \frac{x-5}{x^2-25}$ $\left(\frac{1}{10}\right)$	4. $\lim_{x\to -3} \frac{x+3}{x^2+4x+3}$ $\left(\frac{-1}{2}\right)$
5. $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1}$ $\left(\frac{3}{2}\right)$	6. $\lim_{u\to 1} \frac{u^2-1}{u^3-1}$ $\left(\frac{2}{3}\right)$
7. $\lim_{x\to 0} (2\sin x - 1)$ (-1)	8. $\lim_{x\to 0} \sec x$ (1)
9. $\lim_{x\to 0} (x^2 - 1)(2 - \cos x)$ (-1)	10. $\lim_{x\to \pi} \sqrt{x+4}\ \cos(x+\pi)$ $(\sqrt{\pi+4})$
11. $\lim_{x\to 0} \frac{1+x+\sin x}{3\cos x}$ $\left(\frac{1}{3}\right)$	12. $\lim_{x\to 0} \frac{\sin(1-\cos x)}{1-\cos x}$ (1)
13. $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}$ $\left(\frac{1}{2}\right)$	14. $\lim_{\theta\to 0} \theta\cos\theta$ (0)
15. $\lim_{x\to \pi} \sin(x - \sin x)$ (0)	16. $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$ $\left(\frac{1}{\sqrt{2}}\right)$
17. $\lim_{x\to \infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$ (2)	18. $\lim_{x\to \infty} \left(\frac{1-x^3}{x^2+7x}\right)^5$ (∞)
19. $\lim_{x\to \infty} \frac{2\sqrt{x}+x^{-1}}{3x-7}$ (0)	20. $\lim_{x\to \infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$ (1)
21. $\lim_{x\to \infty} \frac{3-x}{\sqrt{4x^2+25}}$ $(1/2)$	22. $\lim_{x\to \infty} \left(\frac{3}{x^2} - \cos\frac{1}{x}\right)\left(1+\sin\frac{1}{x}\right)$ (-1)