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Lecture Ten

Resonance

10.1 Introduction

This lecture will introduce the very important resonant (or tuned) circuit, which is fundamental to the operation of a wide variety of electrical and electronic systems in use today. The resonant circuit is a combination of R, L, and C elements having a frequency response characteristic similar to the one appearing in Fig. 10.1. Note in the figure that the response is a maximum for the frequency f_r, decreasing to the right and left of this frequency. In other words, for a particular range of frequencies the response will be near or equal to the maximum. The frequencies to the far left or right have very low voltage or current levels and, for all practical purposes, have little effect on the system's response. The radio or television receiver has a response curve for each broadcast station of the type indicated in Fig. 10.1.

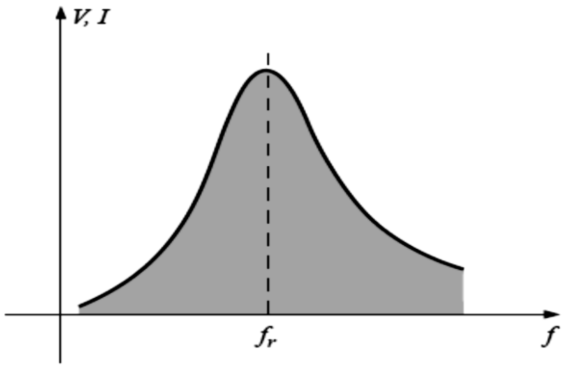


Fig. 10.1 Resonance curve.

10.2 Series Resonant Circuit

A resonant circuit (series or parallel) must have an inductive and a capacitive element. A resistive element will always be present due to the internal resistance of the source (**R_s**), the internal resistance of the inductor (**R_l**), and any added resistance to control the shape of the response curve (**R_{design}**). The basic configuration for the series resonant circuit appears in Fig. 10.2(a) with the resistive elements listed above. The "cleaner" appearance of Fig. 10.2(b) is a result of combining the series resistive elements into one total value. That is,

$$\mathbf{R = R_s + R_l + R_d} \qquad (10.1)$$

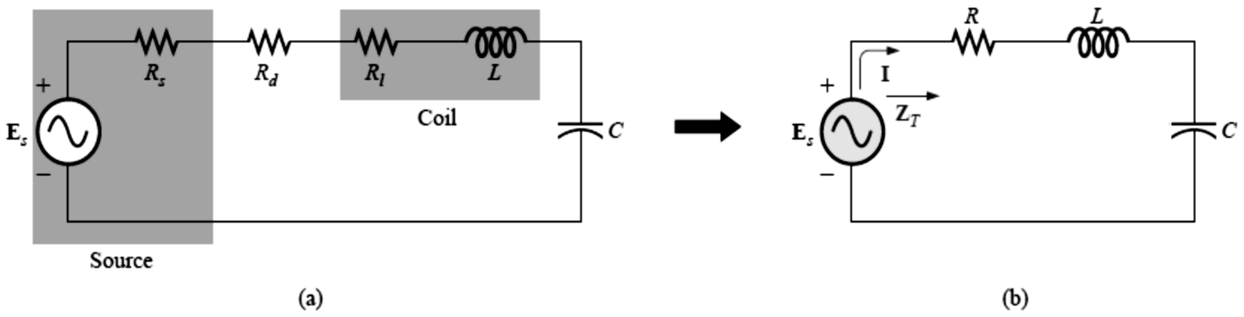


Fig. 10.2 Series resonant circuit.

The total impedance of this network at any frequency is determined by

$$\mathbf{Z_T = R + j\,X_L - j\,X_C = R + j\,(X_L - X_C\,)}$$

The resonant conditions described in the introduction will occur when

$$\mathbf{X_L = X_C} \qquad (10.2)$$

removing the reactive component from the total impedance equation. The total impedance at resonance is then simply

$$\mathbf{Z_{Ts} = R} \qquad (10.3)$$

The resonant frequency can be determined in terms of the inductance and capacitance by examining the defining equation for resonance [Eq. (10.2)]:

$$\boldsymbol{\omega_s = \frac{1}{\sqrt{LC}}} \qquad (10.4)$$

or $$\boldsymbol{fs = \frac{1}{2\pi\sqrt{LC}}} \qquad (10.5)$$

L = henries (H), C = farads (F), f = hertz (Hz)

The current through the circuit at resonance is

$$\boldsymbol{I = \frac{E\angle 0^\circ}{R\angle 0^\circ} = \frac{E}{R}\angle 0^\circ}$$

which you will note is the maximum current for the circuit of Fig. 10.2 for an applied voltage E since Z_T is a minimum value.

The average power to the resistor at resonance is equal to $\mathbf{I^2\,R}$, and the reactive power to the capacitor and inductor are $\mathbf{I^2\,X_C}$ and $\mathbf{I^2\,X_L}$, respectively.

The total apparent power is equal to the average power dissipated by the resistor since $Q_L = Q_C$.

The power factor of the circuit at resonance is

$$\mathbf{pF = \cos\theta = \frac{P}{S} = 1} \tag{10.6}$$

Plotting the power curves of each element on the same set of axes (Fig. 10.3), we note that, even though the total reactive power at any instant is equal to zero.

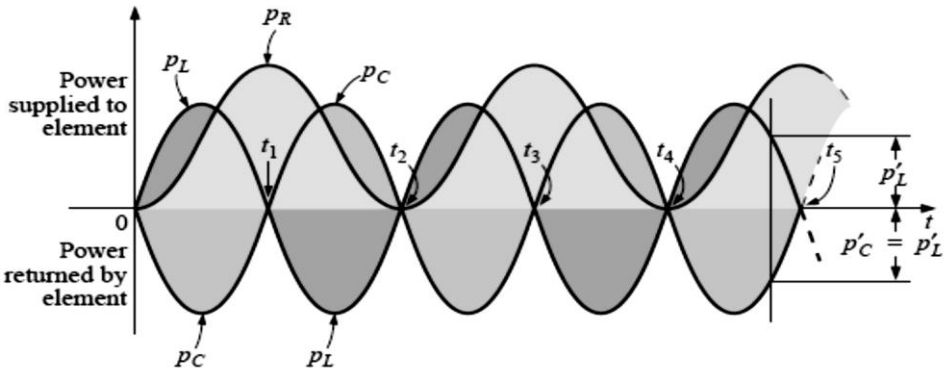


Fig. 10.3 Power curves at resonance for the series resonant circuit.

10.3 The Quality Factor (Q)

The quality factor Q of a series resonant circuit is defined as the ratio of the reactive power of either the inductor or the capacitor to the average power of the resistor at resonance. The quality factor is also an indication of how much energy is placed in storage (continual transfer from one reactive element to the other) compared to that dissipated.

$$Qs = \frac{XL}{R} = \frac{\omega sL}{R} = \frac{1}{\omega_s CR} \tag{10.7}$$

Also $$Qs = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{10.8}$$

By applying the voltage divider rule to the circuit of Fig. 10.2, we obtain

$\mathbf{V_{Ls} = Qs\ E}$

$\mathbf{V_{Cs} = Qs\ E}$

Since Qs is usually greater than 1, the voltage across the capacitor or inductor of a series resonant circuit can be significantly greater than the input voltage.

10.4 Z_T Versus Frequency

The total impedance of the series R-L-C circuit of Fig. 10.2 at any frequency is determined by

$\mathbf{Z_T = R + j\,X_L - j\,X_C}$ **or** $\mathbf{Z_T = R + j\,(X_L - X_C)}$

The magnitude of the impedance Z_T versus frequency is determined by

$$Z_T = \sqrt{[R]^2 + [X_L - X_C]^2}$$

The total-impedance-versus-frequency curve for the series resonant circuit of Fig. 10.2 can be found by applying the impedance-versus- frequency curve for each element of the equation just derived, written in the following form:

$$Z_T(f) = \sqrt{[R(f)]^2 + [X_L(f) - X_C(f)]^2} \quad or \quad Z_T(f) = \sqrt{[R(f)]^2 + [X(f)]^2} \qquad (10.9)$$

where Z_T (f) "means" the total impedance as a function of frequency. For the frequency range of interest, we will assume that the resistance R does not change with frequency. The curve for the inductance, as determined by the reactance equation, is a straight line intersecting the origin with a slope equal to the inductance of the coil. Thus, for the coil,

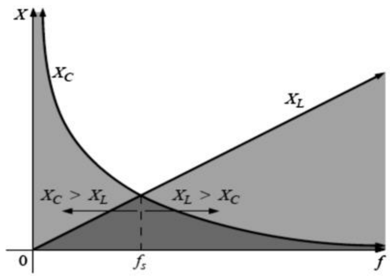


Fig. 10.4 Placing the frequency response of the inductive and capacitive reactance of a series R-L-C circuit on the same set of axes.

$$X_L = 2\pi L . f + 0$$

$$y = a . x + b$$

(where $2\pi L$ is the slope), producing the X_L results is straight line shown in Fig. 20.4.

For the capacitor,

$$X_C = \frac{1}{2\pi f C} \quad \text{or} \quad X_C f = \frac{1}{2\pi C}$$

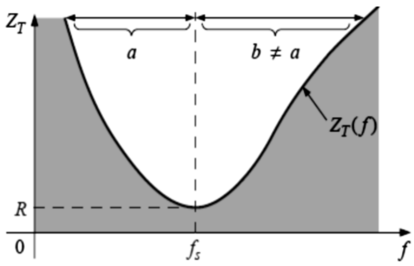


Fig. 10.5 Z_T versus frequency for the series resonant circuit.

which becomes y.x = k, the equation for a hyperbola, where

y (variable) =X_C,   x (variable) = f,   k (constant) = $\frac{1}{2\pi C}$

The hyperbolic curve for $X_C(f)$ is plotted in Fig. 10.4. In particular, note its very large magnitude at low frequencies and its rapid drop- off as the frequency increases.

The condition of resonance is now clearly defined by the point of intersection, where $X_L = X_C$. For frequencies less than fs , it is also quite clear that the network is primarily capacitive ($X_C > X_L$). For frequencies above the resonant condition, $X_L > X_C$, and the network is inductive. Applying eq. (10.9) to the curves of Fig. 10.4, we obtain the curve for Z_T (f) as shown in Fig. 10.5. The minimum impedance occurs at the resonant frequency and is equal to the resistance R. Note that the curve is not symmetrical about the resonant frequency (especially at higher values of Z_T).

The phase angle associated with the total impedance is

$$\theta = \tan^{-1}\frac{(X_L - X_C)}{R} \tag{10.10}$$

At low frequencies, $X_C > X_L$, and v will approach −90° (capacitive), as shown in Fig. 10.6, whereas at high frequencies, $X_L > X_C$, and v will approach 90°In general, therefore, for a series resonant circuit:

f < ***fs***: network capacitive; **I** leads **E**

f > ***fs***: network inductive; **E** leads **I**

f = ***fs***: network resistive; **E** and **I** are in phase.

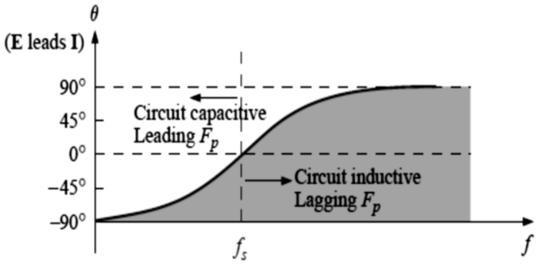


Fig. 10.6 Phase plot for the series resonant circuit.

10.5 Selectivity

If we now plot the magnitude of the current $I = E/Z_T$ versus frequency for a fixed applied voltage E, we obtain the curve shown in Fig. 10.7, which rises from zero to a maximum value of E/R (where Z_T is a minimum) and then drops toward zero (as Z_T increases) at a slower rate than it rose to its peak value. The curve is actually the inverse of the impedance-versus-frequency curve. Since the Z_T curve is not absolutely symmetrical about the resonant frequency, the curve of the current versus frequency has the same property.

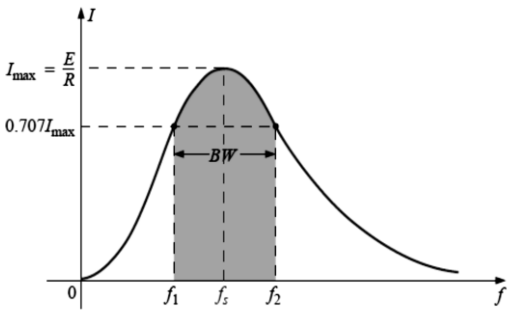


Fig. 10.7 I versus frequency for the series resonant circuit.

There is a definite range of frequencies at which the current is near its maximum value and the impedance is at a minimum. Those frequencies corresponding to 0.707 of the maximum current are called the **band frequencies**, **cutoff frequencies**, or **half-power frequencies**. They are indicated by f_1 and f_2 in Fig. 10.7. The range of frequencies between the two is referred to as the bandwidth (abbreviated **BW**) of the resonant circuit. Half-power frequencies are those frequencies at which the power delivered is one-half that delivered at the resonant frequency; that is,

$$\mathbf{P_{HPF} = \frac{1}{2}\ P_{max}} \qquad (10.11)$$

Since the resonant circuit is adjusted to select a band of frequencies, the curve of Fig. 10.7 is called the **selectivity curve**. The term is derived from the fact that one must be selective in choosing the frequency to ensure that it is in the bandwidth. The smaller the bandwidth, the higher the selectivity. The shape of the curve, as shown in Fig. 10.8, depends on each element of the series R-L-C circuit. Substituting $\sqrt{2}R$ into the equation for the magnitude of Z_T, we find that

$$Z_T = \sqrt{[R]^2 + [X_L - X_C]^2}$$

becomes $\sqrt{2}R = \sqrt{[R]^2 + [X_L - X_C]^2}$

or, squaring both sides, that

$$\mathbf{R^2 = (X_L - X_C)^2 \rightarrow R = X_L - X_C}$$

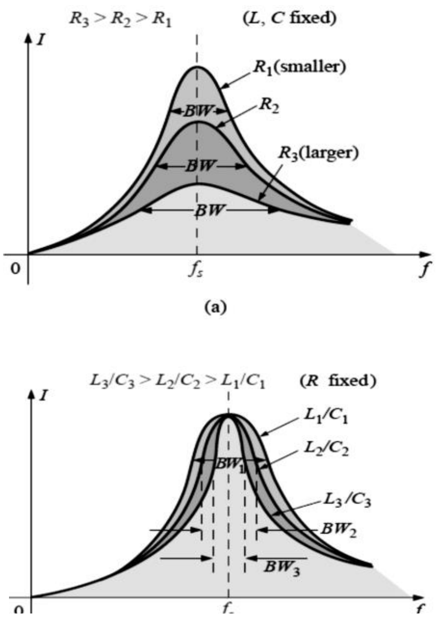


FIG. 10.8
Effect of R, L, and C on the selectivity curve for the series resonant circuit.

Let us first consider the case where $X_L > X_C$, which relates to f_2 or ω_2. Substituting $\omega_2 L$ for X_L and $1/\omega_2 C$ for X_C.

can be written

$$\omega_2^2 - \frac{R}{L}\omega_2 - \frac{1}{LC} = 0$$

Solving the quadratic, we have

$$f_2 = \frac{1}{2\pi}\left[\frac{R}{2L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right] \qquad (10.12)$$

If we repeat the same procedure for $X_C > X_L$, which relates to ω_1 or f_1, the solution f_1 becomes

$$f_1 = \frac{1}{2\pi}\left[-\frac{R}{2L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right] \quad (10.13)$$

The bandwidth (BW) is

$$BW = f_2 - f_1 = \text{Eq. (10.12)} - \text{Eq. (10.13)}$$

and
$$BW = f_2 - f_1 = \frac{R}{2\pi L} \quad (10.14)$$

$$BW = \frac{f_s}{Q_s} \quad (10.15)$$

The ratio BW/fs is sometimes called the fractional bandwidth, providing an indication of the width of the bandwidth compared to the resonant frequency.

$$f_s = \sqrt{f_2 f_1} \quad (10.16)$$

10.6 V_R, V_L, AND V_C

Plotting the magnitude (effective value) of the voltages $\mathbf{V_R}$, $\mathbf{V_L}$, and $\mathbf{V_C}$ and the current I versus frequency for the series resonant circuit on the same set of axes, we obtain the curves shown in Fig. 10.9. Note that the V_R curve has the same shape as the I curve and a peak value equal to the magnitude of the input voltage E. If Q <10 the capacitor max voltage at $f_{Cmax} < f_s$, while the inductor max voltage at $f_{Lmax} > f_s$.

The higher the Qs of the circuit, the closer f_{Cmax} will be to fs, and the closer $V_{Cmax} \cong$ QsE,
and the closer f_{Lmax} will be to fs, and the closer $V_{Lmax} \cong$ QsE,

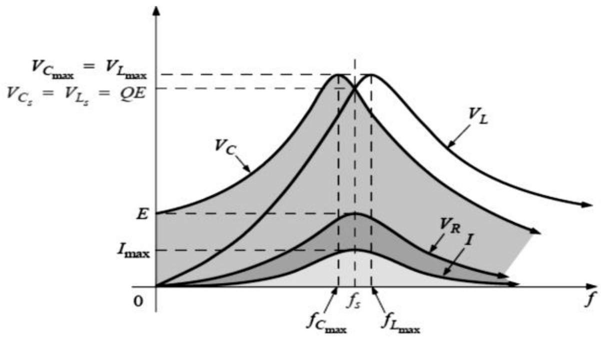


FIG. 10.9 VR, VL, VC, and I versus frequency for a series resonant circuit.

For the condition Qs ≥ 10, the curves of Fig. 10.9 will appear as shown in Fig. 10.10. Note that they each peak (on an approximate basis) at the resonant frequency and have a similar shape.

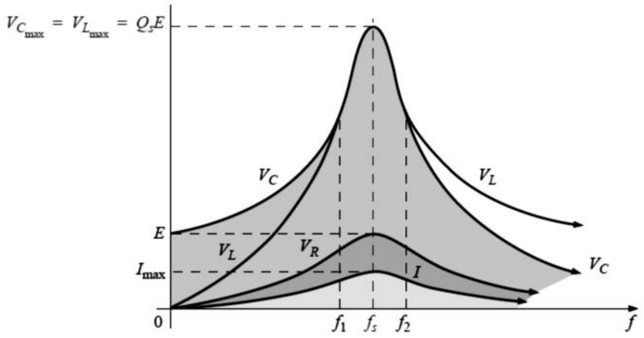


FIG. 10.10 V_R, V_L, V_C, and I for a series resonant circuit where Qs ≥ 10.

In review,

1. V_C and V_L are at their maximum values at or near resonance (depending on Qs).

2. At very low frequencies, V_C is very close to the source voltage and V_L is very close to zero volts, whereas at very high frequencies, V_L approaches the source voltage and V_C approaches zero volts.

3. Both V_R and I peak at the resonant frequency and have the same shape.

10.7 Examples (Series Resonance)

Example 10.1:

a. For the series resonant circuit of Fig. 10.11, find I, V_R, V_L, and V_C at resonance.

b. What is the Qs of the circuit?

c. If the resonant frequency is 5000 Hz, find the bandwidth.

d. What is the power dissipated in the circuit at the half-power frequencies?

Solutions:

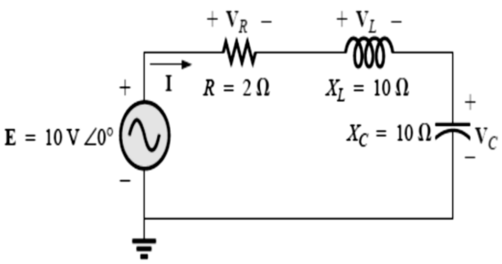


FIG. 10.11 ***Example 10.1.***

a. $Z_{Ts} = R = 2\ \Omega$

$$I = \frac{E}{Z_{Ts}} = 5\ A\ \angle 0°$$

$V_R = E = 10\ V\ \angle 0°$

$V_L = (I\ \angle 0°)(X_L \angle 90°) = (5\ \angle 0°)(10\ \angle\ 90°) = 50\ V\ \angle 90°$

$V_C = (I\ \angle 0°)(X_C \angle -90°) = (5\ \angle 0°)(10\ \angle -90°) = 50\ V\ \angle -90°$

b. $Q_s = \frac{X_L}{R} = \frac{10\Omega}{2\Omega} = 5$

c. $BW = f_2 - f_1 = \frac{f_s}{Q_s} = \frac{5000Hz}{5} = 1000\ Hz$

d. $P_{HPF} = \frac{1}{2} P_{max} = \frac{1}{2} I^2_{max} R = \frac{1}{2}(5\ A)^2(2\ \Omega) = 25\ W$

Example 10.2: The bandwidth of a series resonant circuit is 400 Hz.

a. If the resonant frequency is 4000 Hz, what is the value of Qs?

b. If R = 10 Ω, what is the value of X_L at resonance?

c. Find the inductance L and capacitance C of the circuit.

Solutions:

a. $BW = \frac{f_s}{Q_s}$ or $Qs = \frac{f_s}{BW} = \frac{4000\ Hz}{400\ Hz} = 10$

b. $Qs = \frac{X_L}{R}$ or $XL = QsR = (10)(10\ \Omega) = 100\ \Omega$

c. $X_L = 2\pi f_s L$ or $L = \frac{X_L}{2\pi f_s} = \frac{100\ \Omega}{2\pi(4000\ Hz)} = 3.98\ mH$

$X_C = \frac{1}{2\pi f_s C}$ or $C = \frac{1}{2\pi f_s X_C} = 0.398\ \mu F$

Example 10.3: A series R-L-C circuit has a series resonant frequency of 12,000 Hz.

a. If R = 5 Ω, and if X_L at resonance is 300 Ω, find the bandwidth.

b. Find the cutoff frequencies.

Solutions:

a. $Qs = \frac{X_L}{R} = \frac{300}{5} = 60$

$BW = \frac{f_s}{Q_s} = \frac{12000\ Hz}{60} = 200\ Hz$

b. Since Qs ≥ 10, the bandwidth is bisected by *f*s. Therefore,

$f_2 = fs + \frac{BW}{2} = 12{,}000\ \text{Hz} + 100\ \text{Hz} = 12{,}100\ \text{Hz}$

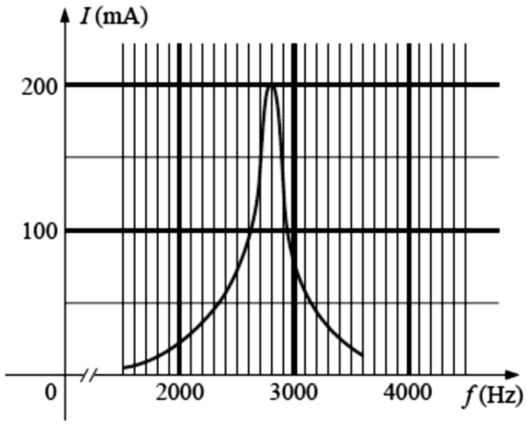
and $f_1 = fs - \frac{BW}{2} = 12{,}000\ \text{Hz} - 100\ \text{Hz} = 11{,}900\ \text{Hz}$

Example 10.4:

a. Determine the Qs and bandwidth for the response curve of Fig. 10.10.

b. For C = 101.5 nF, determine L and R for the series resonant circuit.

c. Determine the applied voltage.



Solutions:

a. The resonant frequency is 2800 Hz. At 0.707 times the peak value,

BW = 200 Hz

and $Qs = \frac{f_s}{BW} = \frac{2800}{200} = 14$

b. $f_s = \frac{1}{2\pi\sqrt{LC}}$ or $L = \frac{1}{4\pi^2 f_s^2 C}$

FIG. 10.12 *Example 10.4.*

$$= \frac{1}{4\pi^2(2.8\times10^3\ Hz)^2(101.5\times10^{-9}\ F)} = 31.832\ \text{mH}$$

$$Qs = \frac{X_L}{R} \quad or \quad R = \frac{X_L}{Q_S} = \frac{2\pi(2800\ Hz)(31.832\times10^{-3}\ H)}{14} = 40\ \Omega$$

c. $I_{max} = E/R$ or $E = I_{max}R = (200\ \text{mA})(40\ \Omega) = 8\ \text{V}$

Example 10.5: A series R-L-C circuit is designed to resonant at $\omega_s = 10^5$ rad/s, have a bandwidth of $0.15\omega_s$, and draw 16 W from a 120-V source at resonance.

a. Determine the value of R.

b. Find the bandwidth in hertz.

c. Find the nameplate values of L and C.

d. Determine the Qs of the circuit.

e. Determine the fractional bandwidth.

Solutions:

a. $P = \frac{E^2}{R}$ and $R = \frac{E^2}{P} = \frac{(120\ V)^2}{16} = 900\ \Omega$

b. $f_s = \frac{\omega_s}{2\pi} = 15{,}915.49\ Hz$

$BW = 0.15f_s = 0.15(15{,}915.49\ \text{Hz}) = 2387.32\ \text{Hz}$

c. Eq. (10.14):

$$BW = \frac{R}{2\pi L} \quad \text{and } L = \frac{R}{2\pi\ BW} = \frac{900\ \Omega}{2\pi(2387.32\ Hz)} = 60\ mH$$

$$f_s = \frac{1}{2\pi\sqrt{LC}} \quad or \quad C = \frac{1}{4\pi^2 f_s^2 L} = \frac{1}{4\pi^2(2387.32\ Hz)^2(60\times10^{-3}\ H)} = 1.67\ \text{nF}$$

d. $Qs = \frac{X_L}{R} = \frac{2\pi f_s L}{R} = \frac{2\pi(15{,}915.49\ Hz)(60\ mH)}{900\ \Omega} = 6.67$

e. $\frac{f_2 - f_1}{f_s} = \frac{BW}{f_s} = \frac{1}{Q_s} = \frac{1}{6.67} = 0.15$

10.8 Parallel Resonant Circuit

The basic format of the series resonant circuit is a series R-L-C combination in series with an applied voltage source. The parallel resonant circuit has the basic configuration of Fig. 10.16, a parallel R-L-C combination in parallel with an applied current source.

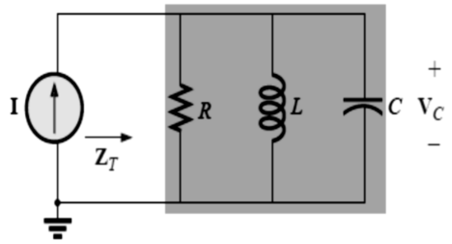


FIG. 10.16
Ideal parallel resonant network.

For the series circuit, the impedance was a minimum at resonance. For the parallel resonant circuit, the impedance is relatively high at resonance. For the network of Fig. 10.16, resonance will occur when $X_L = X_C$, and the resonant frequency will have the same format obtained for series resonance.

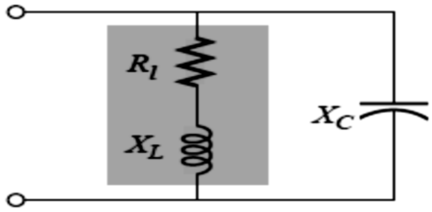


FIG. 10.17
Practical parallel L-C network.

In the practical world, the internal resistance of the coil R_l must be placed in series with the inductor, as shown in Fig. 10.17. Our first effort will be to find a parallel network equivalent (at the terminals) for the series R-L branch of Fig. 10.17. That is,

$$Z_{R\text{-}L} = R_l + j\,X_L \quad \rightarrow \quad Y_{R\text{-}L} = \frac{1}{Rp} + \frac{1}{jX_{LP}}$$

$$\mathbf{Rp} = \frac{R_l^2 + X_L^2}{R_l}, \qquad \mathbf{X_{Lp}} = \frac{R_l^2 + X_L^2}{X_L} \qquad (10.17)$$

as shown in Fig. 10.18.

If we define the parallel combination of R_s and R_p by the notation

$$\mathbf{R = R_s \parallel R_p}$$

the network of Fig. 10.20 will result. It has the same format as the ideal configuration of Fig. 10.16.

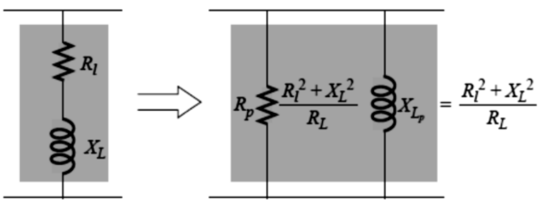
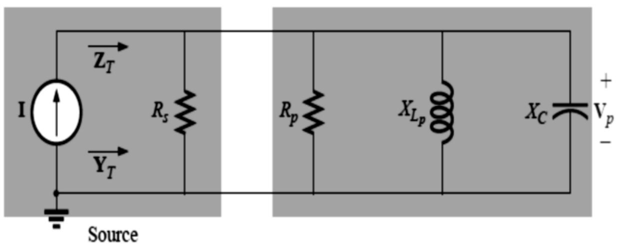


FIG. 10.18 Equivalent parallel network for a series R-L combination.



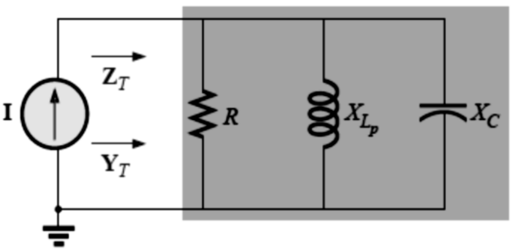


FIG. 10.19 ***Substituting the equivalent parallel network for***

FIG. 10.20 Substituting R = Rs || Rp for

the series R-L combination of Fig. 20.22. ***the network of Fig. 10.19.***

Unity Power Factor, f_p

For the network of Fig. 10.20,

$$Y_T = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} = \frac{1}{R} + j\left(\frac{1}{X_C} - \frac{1}{X_{Lp}}\right) \qquad (10.18)$$

For unity power factor, the reactive component must be zero as defined by

$$\frac{1}{X_C} - \frac{1}{X_{Lp}} = 0$$

Therefore, $X_C = X_{Lp}$ (10.19)

Substituting for X_{Lp} yields

$$\frac{R_l^2 + X_L^2}{X_L} = X_C \qquad (10.20)$$

The resonant frequency, f_p, can now be determined from Eq. (10.20) as follows:

$$f_p = \frac{1}{2\pi\sqrt{LC}}\sqrt{1 - \frac{R_l^2 C}{L}} \qquad (10.21)$$

$$f_p = f_s\sqrt{1 - \frac{R_l^2 C}{L}} \qquad (10.22)$$

where f_p is the resonant frequency of a parallel resonant circuit (for pF = 1) and f_s is the resonant frequency as determined by $X_L = X_C$ for series resonance. Note that unlike a series resonant circuit, the resonant frequency f_p is a function of resistance (in this case R_l) and less than f_s. Recognize also that as the magnitude of R_l approaches zero, f_p rapidly approaches f_s.

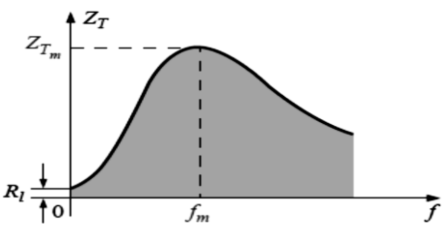
Maximum Impedance, f_m

At $f = f_p$ the input impedance of a parallel resonant circuit will be near its maximum value but not quite its maximum value due to the frequency dependence of Rp. The frequency at which maximum impedance will occur is defined by fm and is slightly more than f_p, as demonstrated in Fig. 10.21. The resulting equation, however, is the following:

$$f_m = f_s\sqrt{1 - \frac{1}{4}\left(\frac{R_l^2 C}{L}\right)} \qquad (10.23)$$

f_m is always closer to f_s and more than f_p. In general,

$$f_s > f_m > f_p$$



Once f_m is determined, the network of Fig. 10.20 can be used to determine the magnitude and phase angle of the total impedance at the resonance condition simply by substituting $f = f_m$ and performing the required calculations. That is,

FIG. 10.21 Z_T versus frequency for the parallel resonant circuit.

$$\mathbf{Z_{Tm} = R \parallel X_{Lp} \parallel X_C} \qquad f = f_m \tag{10.24}$$

10.9 Selectivity Curve for Parallel Resonant Circuits

The Z_T -versus-frequency curve of Fig. 10.21 clearly reveals that a parallel resonant circuit exhibits maximum impedance at resonance (f_m), unlike the series resonant circuit, which experiences minimum resistance levels at resonance. Note also that Z_T is approximately $\mathbf{R_l}$ at f = 0 Hz since $\mathbf{Z_T = R_s \parallel R_l \cong R_l}$.

Since the current I of the current source is constant for any value of Z_T or frequency, the voltage across the parallel circuit will have the same shape as the total impedance Z_T.

$$V_C = V_p = I\,Z_T \tag{10.25}$$

The resonant value of V_C is therefore determined by the value of Z_{Tm} and the magnitude of the current source I. We can speak of the Q of the coil, where

$$Q_{coil} = Q_l = \frac{X_L}{R}$$

The quality factor of the parallel resonant circuit continues to be determined by the ratio of the reactive power to the real power. That is,

$$Qp = \frac{R}{X_{Lp}} = \frac{R}{X_C} \tag{10.26}$$

where $R = R_s \parallel R_p$, and V_p is the voltage across the parallel branches.

For the ideal current source ($R_s = \infty\Omega$) or when R_s is sufficiently large compared to R_p, we can make the following approximation:

$$R = R_s \parallel R_p \cong R_p$$

$$Q_p = \frac{R}{X_{Lp}} = Q_l \qquad R_s >> R_l \tag{10.27}$$

which is simply the quality factor Q_l of the coil.

In general, the bandwidth is still related to the resonant frequency and the quality factor by

$$BW = f_2 - f_1 = \frac{fr}{Qp} \tag{10.28}$$

The cutoff frequencies f1 and f2 can be determined using the equivalent network of Fig. 10.20 and the unity power condition for resonance. The half-power frequencies are defined by the condition that the output voltage is 0.707 times the maximum value.

Setting the input impedance for the network of Fig. 10.20 equal to this value will result in the following relationship:

$$Z = \frac{1}{\frac{1}{R}+j\left(\omega C-\frac{1}{\omega L}\right)} = 0.707R$$

will result in the following after a series of careful mathematical manipulations:

$$f_1 = \frac{1}{4\pi C}\left[\frac{1}{R} - \sqrt{\frac{1}{R^2}+\frac{4C}{L}}\right] \qquad (10.29a)$$

$$f_2 = \frac{1}{4\pi C}\left[\frac{1}{R} + \sqrt{\frac{1}{R^2}+\frac{4C}{L}}\right] \qquad (10.29b)$$

The effect of R_l, L, and C on the shape of the parallel resonance curve, as shown in Fig. 10.22 for the input impedance, is quite similar

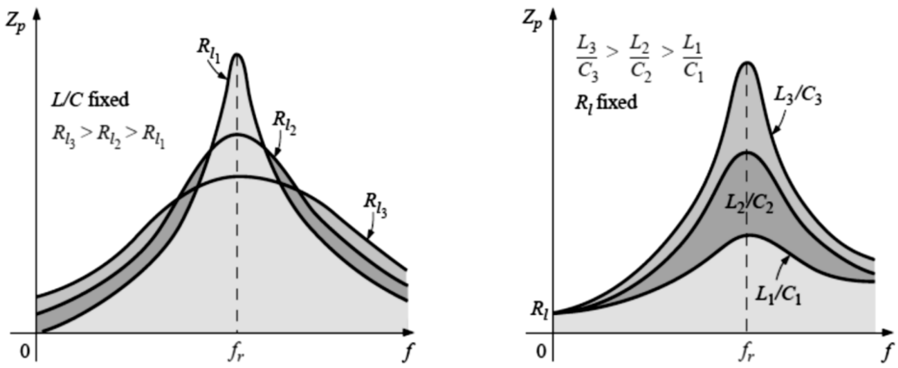


FIG. 10.22 ***Effect of Rl, L, and C on the parallel resonance curve.***

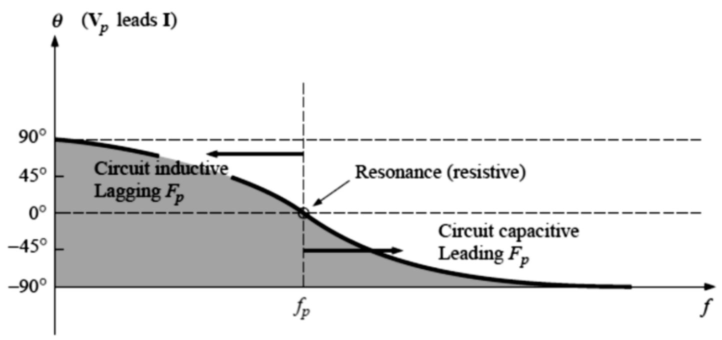


FIG. 10.23 Phase plot for the parallel resonant circuit.

10.10 Effect of $Q_l \geq 10$

The quality factor of the coil $\boldsymbol{Q_l}$ is sufficiently large to permit a number of approximations that simplify the required analysis.

Inductive Reactance, X_{Lp}

$X_{Lp} \cong X_L \qquad Q_l \geq 10$

and since resonance is defined by $X_{Lp} = X_C$, the resulting condition for resonance is reduced to:

$X_L \cong X_C \qquad Q_l \geq 10$

Resonant Frequency, f_p (Unity Power Factor)

$$f_p = f_s\sqrt{1-\frac{1}{Q_l^2}} \qquad Q_l \geq 10$$

$$f_p \cong fs = \frac{1}{2\pi\sqrt{LC}} \qquad Q_l \geq 10$$

Resonant Frequency, f_m (Maximum V_C)

$$f_m = f_s\sqrt{1-\frac{1}{4}\left(\frac{1}{Q_l^2}\right)} \qquad Q_l \geq 10$$

$$f_m \cong fs = \frac{1}{2\pi\sqrt{LC}} \qquad Q_l \geq 10$$

R_p

$$Rp \cong Q_l^2 R_l$$

$$Rp \cong \frac{L}{R_l C} \qquad Q_l \geq 10$$

Z_{Tp}

The total impedance at resonance is now defined by

$$Z_{Tp} = R_s \parallel R_p = R_s \parallel Q_l^2 R_l \qquad Q_l \geq 10$$

$$Z_{Tp} \cong Q_l^2 R_l \qquad Q_l \geq 10 \quad Rs >> Rp$$

Q_p

The quality factor is now defined by

$$Qp = \frac{R}{X_{Lp}} \cong \frac{Rs||Q_l^2 R_l}{X_L}$$

$$Qp \cong Q_l \qquad Q_l \geq 10 \quad Rs >> Rp$$

BW

The bandwidth defined by $\boldsymbol{f_p}$ is

$$BW = f_2 - f_1 = \frac{fp}{Qp} \cong \frac{1}{2\pi}\left[\frac{R_l}{L} + \frac{1}{R_S C}\right]$$

$$BW = f_2 - f_1 \cong \frac{R_l}{2\pi L} \qquad R_s = \infty\Omega$$

I_L and I_C

I_T defined as shown.

$V_C = V_L = V_R = I_T Z_{T_p} = I_T Q_l^2 R_l$

$I_C \cong Q_l I_T \qquad Q_l \geq 10$

$I_L \cong Q_l I_T \qquad Q_l \geq 10$

10.11 Examples (Parallel Resonance)

Example 10.6: Given the parallel network of Fig. 10.24 composed of "ideal" elements:

a. Determine the resonant frequency f_p.

b. Find the total impedance at resonance.

c. Calculate the quality factor, bandwidth, and cutoff frequencies f_1 and f_2 of the system.

d. Find the voltage V_C at resonance.

e. Determine the currents I_L and I_C at resonance.

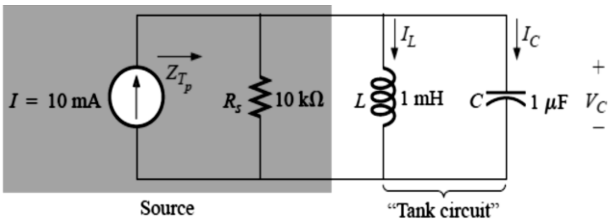


FIG. 10.24 ***Example 10.6.***

Solutions:

a. The fact that R_l is zero ohms results in a very high Q_l $(= X_L/R_l)$, permitting the use of the following equation for f_p:

$$f_p = f_s = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(1\text{ mH})(1\ \mu\text{F})}}$$
$$= \mathbf{5.03\ kHz}$$

b. For the parallel reactive elements:

$$\mathbf{Z}_L \parallel \mathbf{Z}_C = \frac{(X_L \angle 90^\circ)(X_C \angle -90^\circ)}{+j(X_L - X_C)}$$

but $X_L = X_C$ at resonance, resulting in a zero in the denominator of the equation and a very high impedance that can be approximated by an open circuit. Therefore,

$$Z_{T_p} = R_s \parallel \mathbf{Z}_L \parallel \mathbf{Z}_C = R_s = \mathbf{10\ k\Omega}$$

c. $$Q_p = \frac{R_s}{X_{L_p}} = \frac{R_s}{2\pi f_p L} = \frac{10\ \text{k}\Omega}{2\pi(5.03\ \text{kHz})(1\ \text{mH})} = \mathbf{316.41}$$

$$BW = \frac{f_p}{Q_p} = \frac{5.03\ \text{kHz}}{316.41} = \mathbf{15.90\ Hz}$$

Eq. (10.29a):

$$\begin{aligned} f_1 &= \frac{1}{4\pi C}\left[\frac{1}{R} - \sqrt{\frac{1}{R^2} + \frac{4C}{L}}\right] \\ &= \frac{1}{4\pi(1\ \mu\text{F})}\left[\frac{1}{10\ \text{k}\Omega} - \sqrt{\frac{1}{(10\ \text{k}\Omega)^2} + \frac{4(1\ \mu\text{F})}{1\ \text{mH}}}\right] \\ &= \mathbf{5.025\ kHz} \end{aligned}$$

Eq. (10.29b):

$$\begin{aligned} f_2 &= \frac{1}{4\pi C}\left[\frac{1}{R} + \sqrt{\frac{1}{R^2} + \frac{4C}{L}}\right] \\ &= \mathbf{5.041\ kHz} \end{aligned}$$

d. $$V_C = IZ_{T_p} = (10\ \text{mA})(10\ \text{k}\Omega) = \mathbf{100\ V}$$

e. $$I_L = \frac{V_L}{X_L} = \frac{V_C}{2\pi f_p L} = \frac{100\ \text{V}}{2\pi(5.03\ \text{kHz})(1\ \text{mH})} = \frac{100\ \text{V}}{31.6\ \Omega} = \mathbf{3.16\ A}$$

$$I_C = \frac{V_C}{X_C} = \frac{100\ \text{V}}{31.6\ \Omega} = \mathbf{3.16\ A}\ (= Q_p I)$$

<u>Example 10.7</u> For the parallel resonant circuit of Fig. 10.25 with $R_s = \infty\Omega$:

a. Determine f_s, f_m, and f_p, and compare their levels.

b. Calculate the maximum impedance and the magnitude of the voltage V_C at f_m.

c. Determine the quality factor Qp.

d. Calculate the bandwidth.

e. Compare the above results with those obtained using the equations associated with $Q_l \geq 10$.

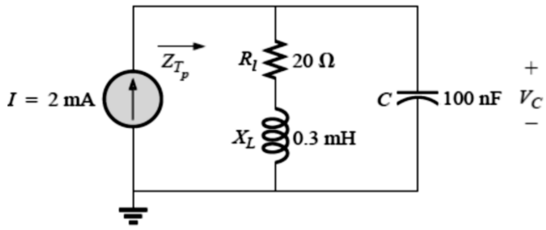


FIG. 10.25 Example 10.7.

Solutions:

a. $f_s = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(0.3 \text{ mH})(100 \text{ nF})}} = \mathbf{29.06 \text{ kHz}}$

$$f_m = f_s\sqrt{1 - \frac{1}{4}\left[\frac{R_l^2 C}{L}\right]}$$

$$= (29.06 \text{ kHz})\sqrt{1 - \frac{1}{4}\left[\frac{(20\ \Omega)^2(100 \text{ nF})}{0.3 \text{ mH}}\right]}$$

$$= \mathbf{28.58 \text{ kHz}}$$

$$f_p = f_s\sqrt{1 - \frac{R_l^2 C}{L}} = (29.06 \text{ kHz})\sqrt{1 - \left[\frac{(20\ \Omega)^2(100 \text{ nF})}{0.3 \text{ mH}}\right]}$$

$$= \mathbf{27.06 \text{ kHz}}$$

Both fm and fp are less than fs, as predicted. In addition, fm is closer to fs than fp, as forecast. fm is about 0.5 kHz less than fs, whereas fp is about 2 kHz less. The differences among fs, fm, and fp suggest a low-Q network.

b. $\mathbf{Z}_{T_m} = (R_l + j X_L) \parallel -j X_C$ at $f = f_m$

$$X_L = 2\pi f_m L = 2\pi(28.58 \text{ kHz})(0.3 \text{ mH}) = 53.87\ \Omega$$

$$X_C = \frac{1}{2\pi f_m C} = \frac{1}{2\pi(28.58 \text{ kHz})(100 \text{ nF})} = 55.69\ \Omega$$

$$R_l + j X_L = 20\ \Omega + j\, 53.87\ \Omega = 57.46\ \Omega \angle 69.63°$$

$$\mathbf{Z}_{T_m} = \frac{(57.46\ \Omega \angle 69.63°)(55.69\ \Omega \angle -90°)}{20\ \Omega + j\, 53.87\ \Omega - j\, 55.69\ \Omega}$$

$$= \mathbf{159.34\ \Omega \angle -15.17°}$$

$$V_{C_{max}} = IZ_{T_m} = (2 \text{ mA})(159.34\ \Omega) = \mathbf{318.68 \text{ mV}}$$

c. $R_s = \infty\ \Omega$; therefore,

$$Q_p = \frac{R_s \parallel R_p}{X_{L_p}} = \frac{R_p}{X_{L_p}} = Q_l = \frac{X_L}{R_l}$$

$$= \frac{2\pi(27.06 \text{ kHz})(0.3 \text{ mH})}{20\ \Omega} = \frac{51\ \Omega}{20\ \Omega} = \mathbf{2.55}$$

The low Q confirms our conclusion of part (a). The differences among fs, fm, and fp will be significantly less for higher-Q networks.

d. $BW = \dfrac{f_p}{Q_p} = \dfrac{27.06\text{ kHz}}{2.55} = \mathbf{10.61\text{ kHz}}$

e. For $Q_l \geq 10$, $f_m = f_p = f_s = \mathbf{29.06\text{ kHz}}$

$$Q_p = Q_l = \frac{2\pi f_s L}{R_l} = \frac{2\pi(29.06\text{ kHz})(0.3\text{ mH})}{20\ \Omega} = \mathbf{2.74}$$

(versus 2.55 above)

$$Z_{T_p} = Q_l^2 R_l = (2.74)^2 \cdot 20\ \Omega = \mathbf{150.15\ \Omega\ \angle 0^\circ}$$

(versus 159.34 Ω ∠−15.17° above)

$$V_{C_{\max}} = IZ_{T_p} = (2\text{ mA})(150.15\ \Omega) = \mathbf{300.3\text{ mV}}$$

(versus 318.68 mV above)

$$BW = \frac{f_p}{Q_p} = \frac{29.06\text{ kHz}}{2.74} = \mathbf{10.61\text{ kHz}}$$

(versus 10.61 kHz above)

<u>Example 10.8</u>: For the network of Fig. 10.26 with *f*p provided:

a. Determine Q_l.

b. Determine Rp.

c. Calculate Z_{Tp}.

d. Find C at resonance.

e. Find Qp.

f. Calculate the BW and cutoff frequencies.

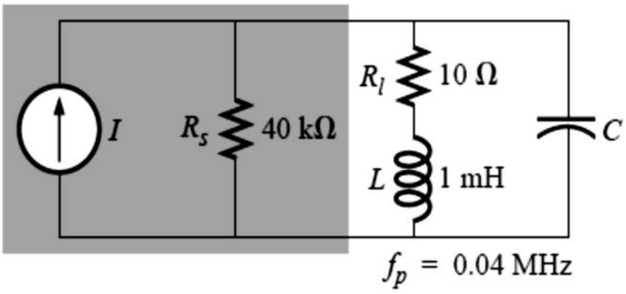


FIG. 10.26 Example 10.8.

Solutions:

a. $Q_l = \dfrac{X_L}{R_l} = \dfrac{2\pi f_p L}{R_l} = \dfrac{2\pi(0.04\text{ MHz})(1\text{ mH})}{10\ \Omega} = \mathbf{25.12}$

b. $Q_l \geq 10$. Therefore,

$$R_p \cong Q_l^2 R_l = (25.12)^2(10\ \Omega) = \mathbf{6.31\text{ k}\Omega}$$

c. $Z_{T_p} = R_s \parallel R_p = 40\text{ k}\Omega \parallel 6.31\text{ k}\Omega = \mathbf{5.45\text{ k}\Omega}$

d. $Q_l \geq 10$. Therefore,

$$f_p \cong \frac{1}{2\pi\sqrt{LC}}$$

and $C = \dfrac{1}{4\pi^2 f^2 L} = \dfrac{1}{4\pi^2(0.04\text{ MHz})^2(1\text{ mH})} = \mathbf{15.83\text{ nF}}$

e. $Q_l \geq 10$. Therefore,

$$Q_p = \frac{Z_{T_p}}{X_L} = \frac{R_s \parallel Q_l^2 R_l}{2\pi f_p L} = \frac{5.45\ \text{k}\Omega}{2\pi(0.04\ \text{MHz})(1\ \text{mH})} = \mathbf{21.68}$$

f. $BW = \dfrac{f_p}{Q_p} = \dfrac{0.04\ \text{MHz}}{21.68} = \mathbf{1.85\ kHz}$

$$f_1 = \frac{1}{4\pi C}\left[\frac{1}{R} - \sqrt{\frac{1}{R^2} + \frac{4C}{L}}\right]$$

$$= \frac{1}{4\pi(15.9\ \text{mF})}\left[\frac{1}{5.45\ \text{k}\Omega} - \sqrt{\frac{1}{(5.45\ \text{k}\Omega)^2} + \frac{4(15.9\ \text{mF})}{1\ \text{mH}}}\right]$$

$$= 5.005 \times 10^6[183.486 \times 10^{-6} - 7.977 \times 10^{-3}]$$

$$= 5.005 \times 10^6[-7.794 \times 10^{-3}]$$

$$= \mathbf{39.009\ kHz} \qquad \text{(ignoring the negative sign)}$$

$$f_2 = \frac{1}{4\pi C}\left[\frac{1}{R} + \sqrt{\frac{1}{R^2} + \frac{4C}{L}}\right]$$

$$= 5.005 \times 10^6[183.486 \times 10^{-6} + 7.977 \times 10^{-3}]$$

$$= 5.005 \times 10^6[8.160 \times 10^{-3}]$$

$$= \mathbf{40.843\ kHz}$$

Note that $f_2 - f_1$ = 40.843 kHz − 39.009 kHz = 1.834 kHz, confirming our solution for the bandwidth above. Note also that the bandwidth is not symmetrical about the resonant frequency, with 991 Hz below and 843 Hz above.

Example 10.9: The equivalent network for the transistor configuration of Fig. 10.27 is provided in Fig. 10.28.

a. Find fp.

b. Determine Qp.

c. Calculate the BW.

d. Determine Vp at resonance.

e. Sketch the curve of VC versus frequency.

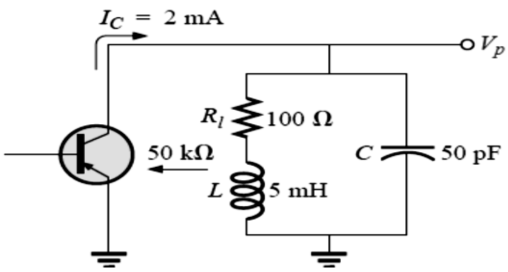


FIG. 10.27 Example 10.9.

Solutions:

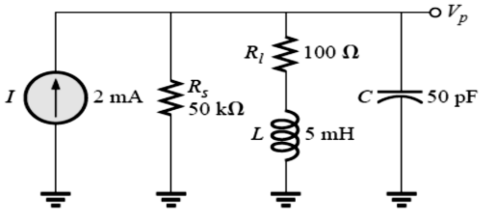


FIG. 10.28
Equivalent network for the transistor configuration of Fig. 10.27.

a. $f_s = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{1}{2\pi\sqrt{(5\text{ mH})(50\text{ pF})}} = 318.31\text{ kHz}$

$X_L = 2\pi f_s L = 2\pi(318.31\text{ kHz})(5\text{ mH}) = 10\text{ k}\Omega$

$Q_l = \dfrac{X_L}{R_l} = \dfrac{10\text{ k}\Omega}{100\ \Omega} = 100 > 10$

Therefore, $fp = fs = 318.31$ kHz. Using Eq. (20.31) would result in ≅318.5 kHz.

b. $Q_p = \dfrac{R_s \parallel R_p}{X_L}$

$R_p = Q_l^2 R_l = (100)^2 100\ \Omega = 1\text{ M}\Omega$

$Q_p = \dfrac{50\text{ k}\Omega \parallel 1\text{ M}\Omega}{10\text{ k}\Omega} = \dfrac{47.62\text{ k}\Omega}{10\text{ k}\Omega} = \mathbf{4.76}$

Note the drop in Q from $Q_l = 100$ to $Q_p = 4.76$ due to R_s..

c. $BW = \dfrac{f_p}{Q_p} = \dfrac{318.31\text{ kHz}}{4.76} = \mathbf{66.87\text{ kHz}}$

On the other hand,

$$BW = \frac{1}{2\pi}\left(\frac{R_l}{L} + \frac{1}{R_s C}\right) = \frac{1}{2\pi}\left[\frac{100\ \Omega}{5\text{ mH}} + \frac{1}{(50\text{ k}\Omega)(50\text{ pF})}\right]$$
$$= \mathbf{66.85\text{ kHz}}$$

compares very favorably with the above solution.

d. $V_p = I Z_{Tp} = (2\text{ mA})(R_s \parallel R_p) = (2\text{ mA})(47.62\text{ k}\Omega)$

= 95.24 V

e. See Fig. 10.29.

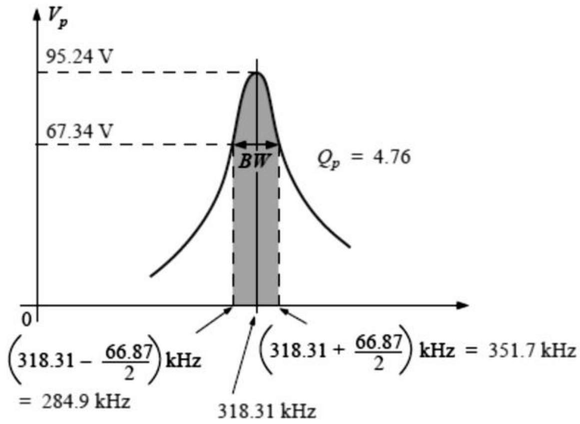


FIG. 10.29 ***Resonance curve for the network of Fig. 10.28.***

Example 10.10: Repeat Example 10.9, but ignore the effects of Rs, and compare results.

Solutions:

a. *f*p is the same, 318.31 kHz.

b. For Rs = ∞ Ω,

$$Q_p = Q_l = \mathbf{100} \qquad \text{(versus 4.76)}$$

c. $BW = \dfrac{f_p}{Q_p} = \dfrac{318.31\text{ kHz}}{100} = \mathbf{3.183\ kHz} \qquad \text{(versus 66.87 kHz)}$

d. $Z_{T_p} = R_p = 1\text{ M}\Omega \qquad \text{(versus 47.62 k}\Omega)$

$V_p = IZ_{T_p} = (2\text{ mA})(1\text{ M}\Omega) = \mathbf{2000\ V} \qquad \text{(versus 95.24 V)}$

The results obtained clearly reveal that the source resistance can have a significant impact on the response characteristics of a parallel resonant circuit.

Example 10.11: Design a parallel resonant circuit to have the response curve of Fig. 10.30 using a 1-mH, 10-Ω inductor and a current source with an internal resistance of 40 kΩ.

Solution:

BW _fp_Qp

Therefore,

$$Q_p = \frac{f_p}{BW} = \frac{50{,}000\text{ Hz}}{2500\text{ Hz}} = \mathbf{20}$$

$$X_L = 2\pi f_p L = 2\pi(50\text{ kHz})(1\text{ mH}) = 314\ \Omega$$

and $$Q_l = \frac{X_L}{R_l} = \frac{314\ \Omega}{10\ \Omega} = \mathbf{31.4}$$

$$R_p = Q_l^2 R = (31.4)^2(10\ \Omega) = \mathbf{9859.6\ \Omega}$$

$$Q_p = \frac{R}{X_L} = \frac{R_s \parallel 9859.6\ \Omega}{314\ \Omega} = 20 \qquad \text{(from above)}$$

so that $$\frac{(R_s)(9859.6)}{R_s + 9859.6} = 6280$$

resulting in $$R_s = 17.298\text{ k}\Omega$$

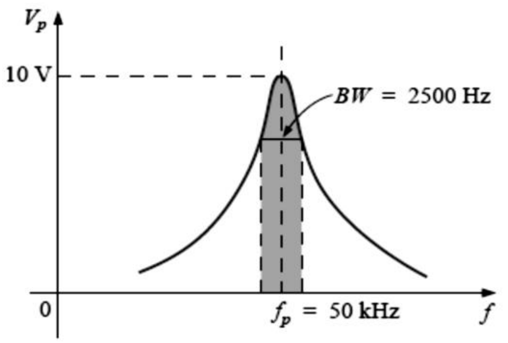


FIG. 10.30 ***Example 10.11.***

However, the source resistance was given as 40 kΩ. We must therefore add a parallel resistor (R′) that will reduce the 40 kΩ to approximately 17.298 kΩ; that is,

$$\frac{(40\ \mathrm{k\Omega})(R')}{40\ \mathrm{k\Omega} + R'} = 17.298\ \mathrm{k\Omega}$$

Solving for R′:

$$R' = 30.481\ \mathrm{k\Omega}$$

The closest commercial value is 30 kΩ. At resonance, $X_L = X_C$, and

$$X_C = \frac{1}{2\pi f_p C}$$

$$C = \frac{1}{2\pi f_p X_C} = \frac{1}{2\pi(50\ \mathrm{kHz})(314\ \Omega)}$$

and $$C \cong \mathbf{0.01\ \mu F} \qquad \text{(commercially available)}$$

$$Z_{T_p} = R_s \parallel Q_l^2 R_l$$
$$= 17.298\ \mathrm{k\Omega} \parallel 9859.6\ \Omega$$
$$= 6.28\ \mathrm{k\Omega}$$

with $$V_p = IZ_{T_p}$$

and $$I = \frac{V_p}{Z_{T_p}} = \frac{10\ \mathrm{V}}{6.28\ \mathrm{k\Omega}} \cong \mathbf{1.6\ mA}$$

The network appears in Fig. 10.31.

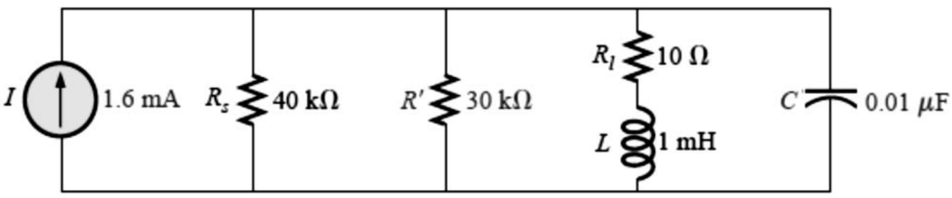


FIG. 10.31 ***Network designed to meet the criteria of Fig. 10.30.***