2/5 Normal and Tangential Coordinates (n-t)

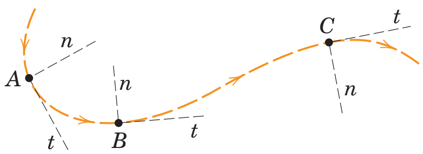


Figure 2/9

As we mentioned in Art. 2/1, one of the common descriptions of curvilinear motion uses path variables, which are measurements made along the tangent t and normal n to the path of the particle. These coordinates provide a very natural description for curvilinear motion and are frequently the most direct and convenient coordinates to use. The n- and t-coordinates are considered to move along the path with the particle, as seen in Fig. 2/9 where the particle advances from A to B to C. The positive direction for n at any position is always taken toward the center of curvature of the path. As seen from Fig. 2/9, the positive n-direction will shift from one side of the curve to the other side if the curvature changes direction.

Velocity and Acceleration

We now use the coordinates n and t to describe the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ which were introduced in Art. 2/3 for the curvilinear motion of a particle. For this purpose, we introduce unit vectors $\mathbf{e}_n$ in the n-direction and $\mathbf{e}_t$ in the t-direction, as shown in Fig. 2/10a for the position of the particle at point A on its path. During a differential increment of time dt, the particle moves a differential distance ds along the curve from A to A'. With the radius of curvature of the path at this position designated by ρ, we see that $ds = \rho\, d\beta$, where β is in radians. It is unnecessary to consider the differential change in ρ between A and A' because a higher-order term would be introduced which disappears in the limit. Thus, the magnitude of the velocity can be written $v = ds/dt = \rho\, d\beta/dt$, and we can write the velocity as the vector

$$\mathbf{v} = v\mathbf{e}_t = \rho\dot{\beta}\mathbf{e}_t \tag{2/7}$$

The acceleration $\mathbf{a}$ of the particle was defined in Art. 2/3 as $\mathbf{a} = d\mathbf{v}/dt$, and we observed from Fig. 2/5 that the acceleration is a vector which reflects both the change in magnitude and the change in direction of $\mathbf{v}$. We now differentiate $\mathbf{v}$ in Eq. 2/7 by applying the ordinary rule for the differentiation of the product of a scalar and a vector* and get

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\mathbf{e}_t)}{dt} = v\dot{\mathbf{e}}_t + \dot{v}\mathbf{e}_t \tag{2/8}$$

where the unit vector $\mathbf{e}_t$ now has a nonzero derivative because its direction changes.

To find $\dot{\mathbf{e}}_t$ we analyze the change in $\mathbf{e}_t$ during a differential increment of motion as the particle moves from A to A' in Fig. 2/10a. The unit vector $\mathbf{e}_t$ correspondingly changes to $\mathbf{e}_t'$, and the vector difference $d\mathbf{e}_t$ is shown in part b of the figure. The vector $d\mathbf{e}_t$ in the limit has a magnitude equal to the length of the arc $|\mathbf{e}_t|\, d\beta = d\beta$ obtained by swinging the unit vector $\mathbf{e}_t$ through the angle $d\beta$ expressed in radians.

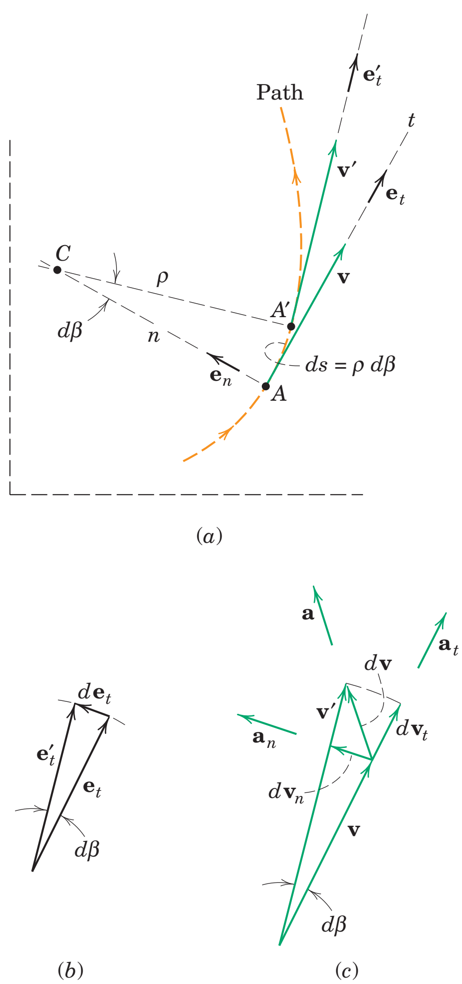


Figure 2/10

*See Art. C/7 of Appendix C.

The direction of $d\mathbf{e}_t$ is given by $\mathbf{e}_n$. Thus, we can write $d\mathbf{e}_t = \mathbf{e}_n \, d\beta$. Dividing by $d\beta$ gives

$$\frac{d\mathbf{e}_t}{d\beta} = \mathbf{e}_n$$

Dividing by dt gives $d\mathbf{e}_t/dt = (d\beta/dt)\mathbf{e}_n$, which can be written

$$\dot{\mathbf{e}}_t = \dot{\beta}\mathbf{e}_n \tag{2/9}$$

With the substitution of Eq. 2/9 and $\dot{\beta}$ from the relation $v = \rho\dot{\beta}$, Eq. 2/8 for the acceleration becomes

$$\mathbf{a} = \frac{v^2}{\rho}\mathbf{e}_n + \dot{v}\mathbf{e}_t \tag{2/10}$$

where

$$a_n = \frac{v^2}{\rho} = \rho\dot{\beta}^2 = v\dot{\beta}$$

$$a_t = \dot{v} = \ddot{s}$$

$$a = \sqrt{a_n{}^2 + a_t{}^2}$$

We stress that $a_t = \dot{v}$ is the time rate of change of the speed v. Finally, we note that $a_t = \dot{v} = d(\rho\dot{\beta})/dt = \rho\ddot{\beta} + \dot{\rho}\dot{\beta}$. This relation, however, finds little use because we seldom have reason to compute $\dot{\rho}$.

Geometric Interpretation

Full understanding of Eq. 2/10 comes only when we clearly see the geometry of the physical changes it describes. Figure 2/10*c* shows the velocity vector $\mathbf{v}$ when the particle is at A and $\mathbf{v}'$ when it is at A'. The vector change in the velocity is $d\mathbf{v}$, which establishes the direction of the acceleration $\mathbf{a}$. The n-component of $d\mathbf{v}$ is labeled $d\mathbf{v}_n$, and in the limit its magnitude equals the length of the arc generated by swinging the vector $\mathbf{v}$ as a radius through the angle $d\beta$. Thus, $|d\mathbf{v}_n| = v \, d\beta$ and the n-component of acceleration is $a_n = |d\mathbf{v}_n|/dt = v(d\beta/dt) = v\dot{\beta}$ as before. The t-component of $d\mathbf{v}$ is labeled $d\mathbf{v}_t$, and its magnitude is simply the change dv in the magnitude or length of the velocity vector. Therefore, the t-component of acceleration is $a_t = dv/dt = \dot{v} = \ddot{s}$ as before. The acceleration vectors resulting from the corresponding vector changes in velocity are shown in Fig. 2/10*c*.

It is especially important to observe that the normal component of acceleration a_n is *always directed toward the center of curvature C*. The tangential component of acceleration, on the other hand, will be in the positive t-direction of motion if the speed v is increasing and in the negative t-direction if the speed is decreasing. In Fig. 2/11 are shown schematic representations of the variation in the acceleration vector for a particle moving from A to B with (*a*) increasing speed and (*b*) decreasing speed. At an inflection point on the curve, the normal acceleration v^2/ρ goes to zero because ρ becomes infinite.

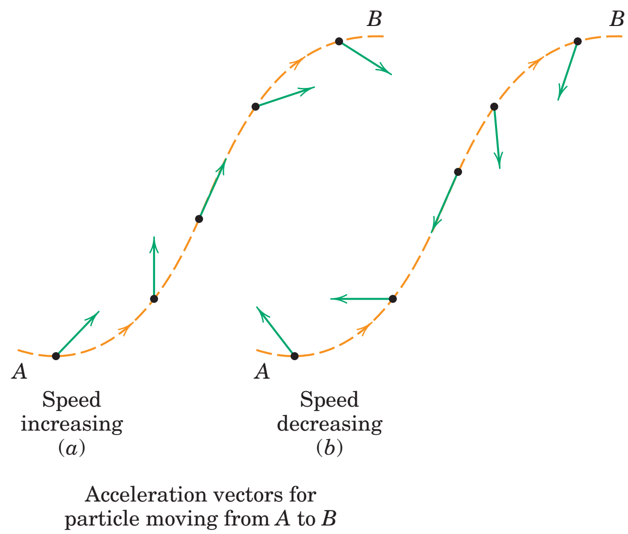


Figure 2/11

Circular Motion

Circular motion is an important special case of plane curvilinear motion where the radius of curvature ρ becomes the constant radius r of the circle and the angle β is replaced by the angle θ measured from any convenient radial reference to OP, Fig. 2/12. The velocity and the acceleration components for the circular motion of the particle P become

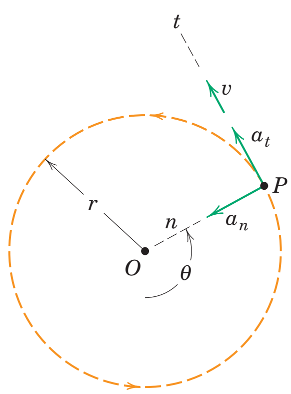


Figure 2/12

$$
\begin{aligned}
v &= r\dot{\theta} \\
a_n &= v^2/r = r\dot{\theta}^2 = v\dot{\theta} \\
a_t &= \dot{v} = r\ddot{\theta}
\end{aligned}
\qquad \textbf{(2/11)}
$$

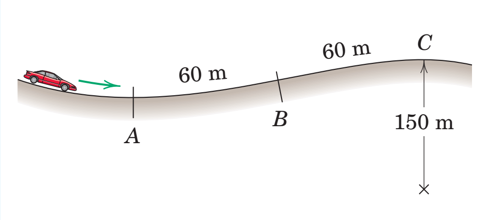
We find repeated use for Eqs. 2/10 and 2/11 in dynamics, so these relations and the principles behind them should be mastered.

GaryTramontina/Bloomberg viaGetty Images

An example of uniform circular motion is this car moving with constant speed around a wet skidpad (a circular roadway with a diameter of about 200 feet).

SAMPLE PROBLEM 2/7

To anticipate the dip and hump in the road, the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100 km/h at the bottom A of the dip and 50 km/h at the top C of the hump, which is 120 m along the road from A. If the passengers experience a total acceleration of 3 m/s^2 at A and if the radius of curvature of the hump at C is 150 m, calculate (*a*) the radius of curvature ρ at A, (*b*) the acceleration at the inflection point B, and (*c*) the total acceleration at C.



Solution. The dimensions of the car are small compared with those of the path, so we will treat the car as a particle. The velocities are ①

$$v_A = \left(100\,\frac{\text{km}}{\text{h}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right)\left(1000\,\frac{\text{m}}{\text{km}}\right) = 27.8\text{ m/s}$$

$$v_C = 50\,\frac{1000}{3600} = 13.89\text{ m/s}$$

We find the constant deceleration along the path from

$$\left[\int v\,dv = \int a_t\,ds\right] \qquad \int_{v_A}^{v_C} v\,dv = a_t \int_0^s ds$$

$$a_t = \frac{1}{2s}(v_C{}^2 - v_A{}^2) = \frac{(13.89)^2 - (27.8)^2}{2(120)} = -2.41\text{ m/s}^2$$

(a) Condition at A. With the total acceleration given and a_t determined, we can easily compute a_n and hence ρ from

$[a^2 = a_n{}^2 + a_t{}^2] \qquad a_n{}^2 = 3^2 - (2.41)^2 = 3.19 \qquad a_n = 1.785\text{ m/s}^2$

$[a_n = v^2/\rho] \qquad \rho = v^2/a_n = (27.8)^2/1.785 = 432\text{ m}$ *Ans.*

(b) Condition at B. Since the radius of curvature is infinite at the inflection point, $a_n = 0$ and

$$a = a_t = -2.41\text{ m/s}^2$$ *Ans.*

(c) Condition at C. The normal acceleration becomes

$[a_n = v^2/\rho] \qquad a_n = (13.89)^2/150 = 1.286\text{ m/s}^2$

With unit vectors $\mathbf{e}_n$ and $\mathbf{e}_t$ in the n- and t-directions, the acceleration may be written

$$\mathbf{a} = 1.286\mathbf{e}_n - 2.41\mathbf{e}_t\text{ m/s}^2$$

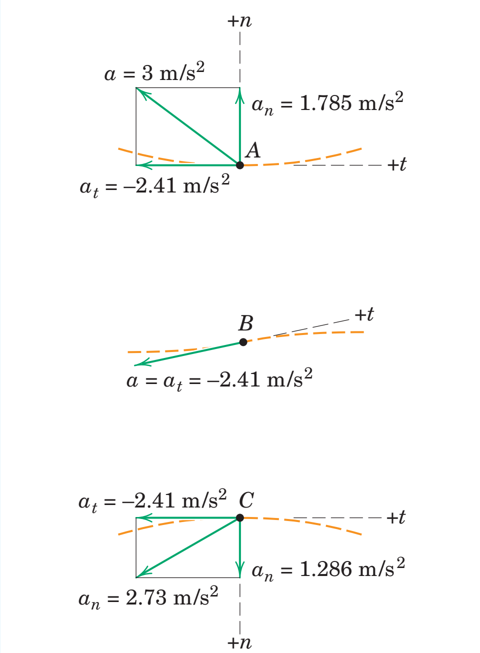
where the magnitude of $\mathbf{a}$ is

$[a = \sqrt{a_n{}^2 + a_t{}^2}] \qquad a = \sqrt{(1.286)^2 + (-2.41)^2} = 2.73\text{ m/s}^2$ *Ans.*

The acceleration vectors representing the conditions at each of the three points are shown for clarification.

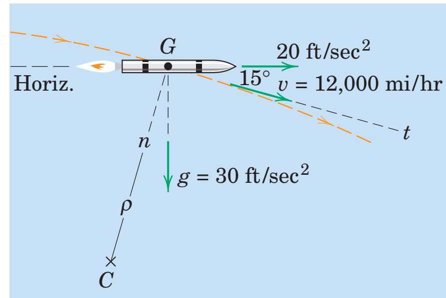
Helpful Hint

① Actually, the radius of curvature to the road differs by about 1 m from that to the path followed by the center of mass of the passengers, but we have neglected this relatively small difference.



SAMPLE PROBLEM 2/8

A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of 20 ft/sec^2, and the downward acceleration component is the acceleration due to gravity at that altitude, which is $g = 30$ ft/sec^2. At the instant represented, the velocity of the mass center G of the rocket along the 15° direction of its trajectory is 12,000 mi/hr. For this position determine (a) the radius of curvature of the flight trajectory, (b) the rate at which the speed v is increasing, (c) the angular rate $\dot{\beta}$ of the radial line from G to the center of curvature C, and (d) the vector expression for the total acceleration $\mathbf{a}$ of the rocket.



Solution. We observe that the radius of curvature appears in the expression for the normal component of acceleration, so we use n- and t-coordinates to describe the motion of G. The n- and t-components of the total acceleration are obtained by resolving the given horizontal and vertical accelerations into their n- and t-components and then combining. From the figure we get ①

$$a_n = 30 \cos 15° - 20 \sin 15° = 23.8 \text{ ft/sec}^2$$

$$a_t = 30 \sin 15° + 20 \cos 15° = 27.1 \text{ ft/sec}^2$$

(a) We may now compute the radius of curvature from

② $[a_n = v^2/\rho]$ $\qquad \rho = \dfrac{v^2}{a_n} = \dfrac{[(12{,}000)(44/30)]^2}{23.8} = 13.01(10^6)$ ft $\qquad$ *Ans.*

(b) The rate at which v is increasing is simply the t-component of acceleration.

$[\dot{v} = a_t]$ $\qquad \dot{v} = 27.1$ ft/sec^2 $\qquad$ *Ans.*

(c) The angular rate $\dot{\beta}$ of line GC depends on v and ρ and is given by

$[v = \rho\dot{\beta}]$ $\qquad \dot{\beta} = v/\rho = \dfrac{12{,}000(44/30)}{13.01(10^6)} = 13.53(10^{-4})$ rad/sec $\qquad$ *Ans.*

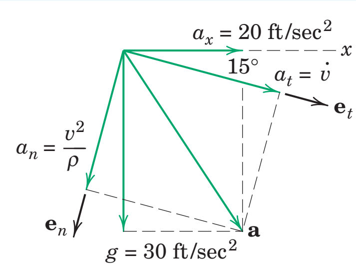
(d) With unit vectors $\mathbf{e}_n$ and $\mathbf{e}_t$ for the n- and t-directions, respectively, the total acceleration becomes

$$\mathbf{a} = 23.8\mathbf{e}_n + 27.1\mathbf{e}_t \text{ ft/sec}^2 \qquad \textit{Ans.}$$

Helpful Hints

① Alternatively, we could find the resultant acceleration and then resolve it into n- and t-components.

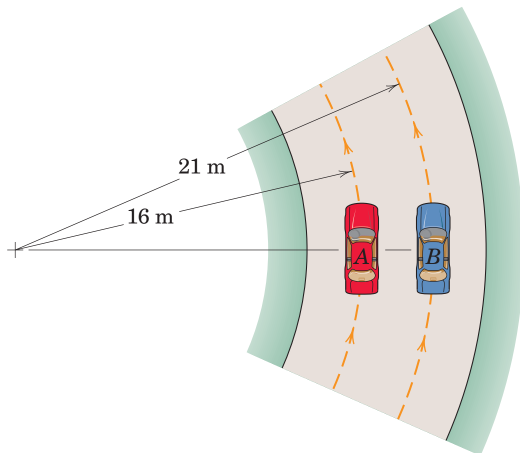
② To convert from mi/hr to ft/sec, multiply by $\dfrac{5280 \text{ ft/mi}}{3600 \text{ sec/hr}} = \dfrac{44 \text{ ft/sec}}{30 \text{ mi/hr}}$ which is easily remembered, as 30 mi/hr is the same as 44 ft/sec.



PROBLEMS

Introductory Problems

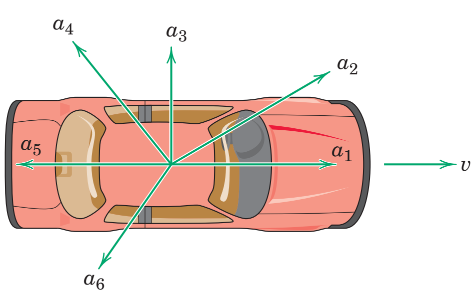
2/97 Determine the maximum speed for each car if the normal acceleration is limited to $0.88g$. The roadway is unbanked and level.



Problem 2/97

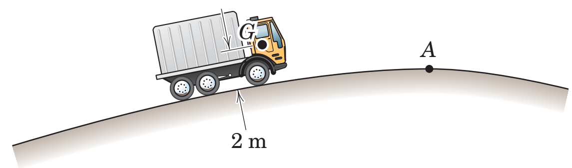
2/98 A car is traveling around a circular track of 800-ft radius. If the magnitude of its total acceleration is 10 ft/sec^2 at the instant when its speed is 45 mi/hr, determine the rate at which the car is changing its speed.

2/99 Six acceleration vectors are shown for the car whose velocity vector is directed forward. For each acceleration vector describe in words the instantaneous motion of the car.



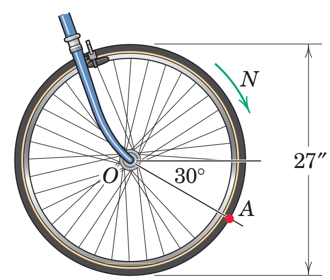
Problem 2/99

2/100 The driver of the truck has an acceleration of $0.4g$ as the truck passes over the top A of the hump in the road at constant speed. The radius of curvature of the road at the top of the hump is 98 m, and the center of mass G of the driver (considered a particle) is 2 m above the road. Calculate the speed v of the truck.



Problem 2/100

2/101 A bicycle is placed on a service rack with its wheels hanging free. As part of a bearing test, the front wheel is spun at the rate $N = 45$ rev/min. Assume that this rate is constant and determine the speed v and magnitude a of the acceleration of point A.

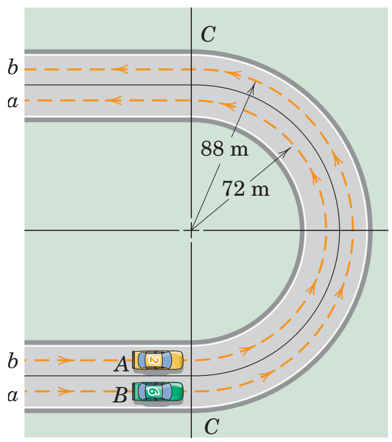


Problem 2/101

2/102 A ship which moves at a steady 20-knot speed (1 knot = 1.852 km/h) executes a turn to port by changing its compass heading at a constant counterclockwise rate. If it requires 60 s to alter course 90°, calculate the magnitude of the acceleration **a** of the ship during the turn.

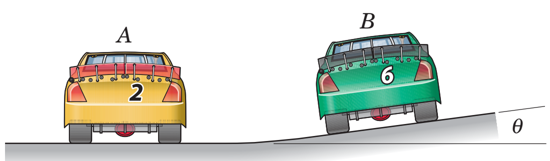
2/103 A train enters a curved horizontal section of track at a speed of 100 km/h and slows down with constant deceleration to 50 km/h in 12 seconds. An accelerometer mounted inside the train records a horizontal acceleration of 2 m/s^2 when the train is 6 seconds into the curve. Calculate the radius of curvature ρ of the track for this instant.

2/104 The two cars A and B enter an unbanked and level turn. They cross line C-C simultaneously, and each car has the speed corresponding to a maximum normal acceleration of $0.9g$ in the turn. Determine the elapsed time for each car between its two crossings of line C-C. What is the relative position of the two cars as the second car exits the turn? Assume no speed changes throughout.



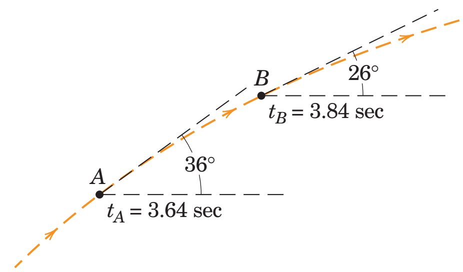
Problem 2/104

2/105 Revisit the two cars of the previous problem, only now the track has variable banking—a concept shown in the figure. Car A is on the unbanked portion of the track and its normal acceleration remains at $0.9g$. Car B is on the banked portion of the track and its normal acceleration is limited to $1.12g$. If the cars approach line C-C with speeds equal to the respective maxima in the turn, determine the time for each car to negotiate the turn as delimited by line C-C. What is the relative position of the two cars as the second car exits the turn? Assume no speed changes throughout.



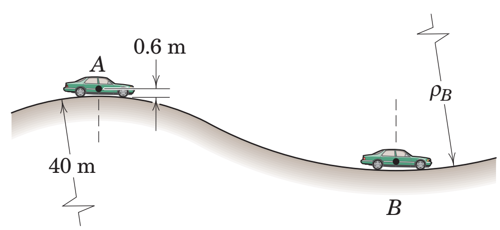
Problem 2/105

2/106 A particle moves along the curved path shown. If the particle has a speed of 40 ft/sec at A at time t_A and a speed of 44 ft/sec at B at time t_B, determine the average values of the acceleration of the particle between A and B, both normal and tangent to the path.



Problem 2/106

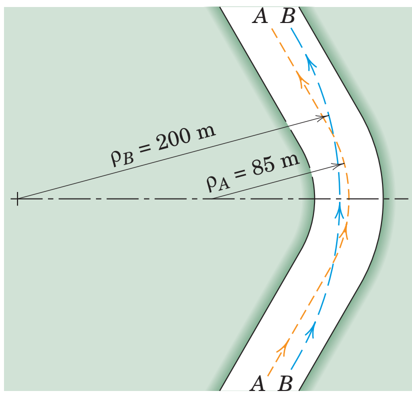
2/107 The speed of a car increases uniformly with time from 50 km/h at A to 100 km/h at B during 10 seconds. The radius of curvature of the hump at A is 40 m. If the magnitude of the total acceleration of the mass center of the car is the same at B as at A, compute the radius of curvature ρ_B of the dip in the road at B. The mass center of the car is 0.6 m from the road.



Problem 2/107

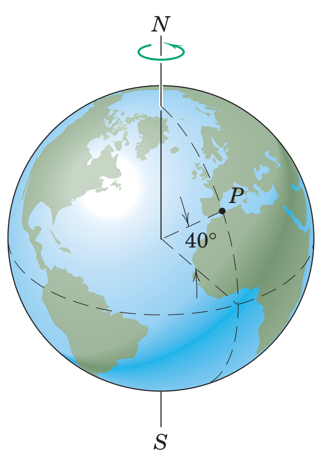
Representative Problems

2/108 The figure shows two possible paths for negotiating an unbanked turn on a horizontal portion of a race course. Path *A-A* follows the centerline of the road and has a radius of curvature $\rho_A = 85$ m, while path *B-B* uses the width of the road to good advantage in increasing the radius of curvature to $\rho_B = 200$ m. If the drivers limit their speeds in their curves so that the lateral acceleration does not exceed $0.8g$, determine the maximum speed for each path.



Problem 2/108

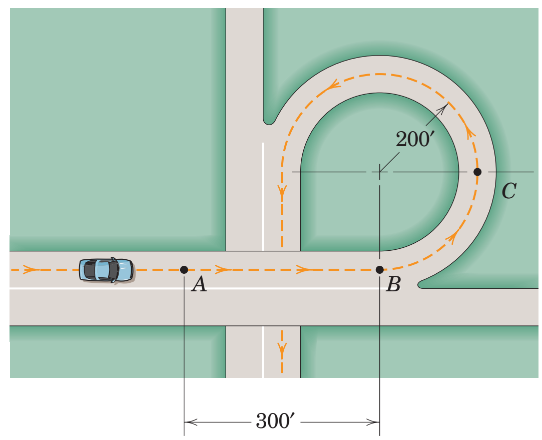
2/109 Consider the polar axis of the earth to be fixed in space and compute the magnitudes of the velocity and acceleration of a point *P* on the earth's surface at latitude 40° north. The mean diameter of the earth is 12 742 km and its angular velocity is $0.7292(10^{-4})$ rad/s.



Problem 2/109

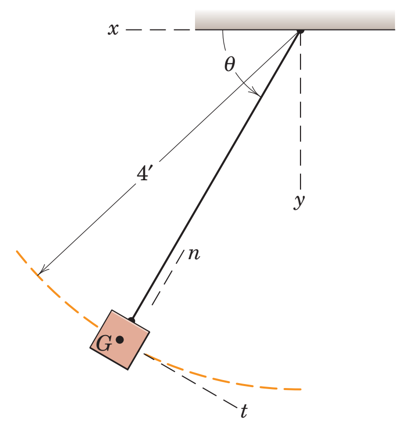
2/110 A satellite travels with constant speed v in a circular orbit 320 km above the earth's surface. Calculate v knowing that the acceleration of the satellite is the gravitational acceleration at its altitude. (*Note:* Review Art. 1/5 as necessary and use the mean value of g and the mean value of the earth's radius. Also recognize that v is the magnitude of the velocity of the satellite with respect to the center of the earth.)

2/111 The car is traveling at a speed of 60 mi/hr as it approaches point *A*. Beginning at *A*, the car decelerates at a constant 7 ft/sec^2 until it gets to point *B*, after which its constant rate of decrease of speed is 3 ft/sec^2 as it rounds the interchange ramp. Determine the magnitude of the total car acceleration (*a*) just before it gets to *B*, (*b*) just after it passes *B*, and (*c*) at point *C*.



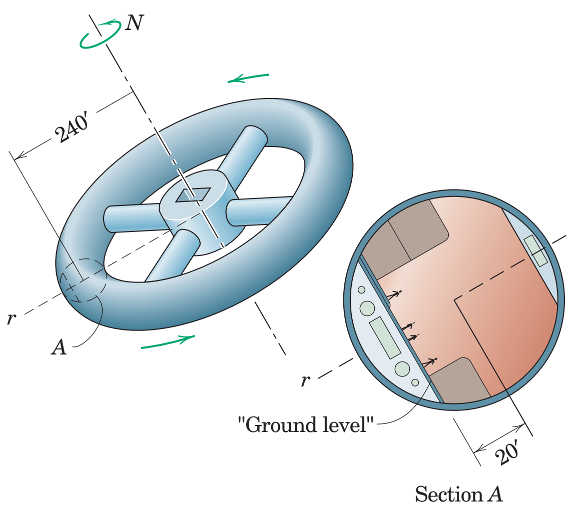
Problem 2/111

2/112 Write the vector expression for the acceleration **a** of the mass center *G* of the simple pendulum in both *n-t* and *x-y* coordinates for the instant when $\theta = 60°$ if $\dot{\theta} = 2$ rad/sec and $\ddot{\theta} = 4.025$ rad/sec^2.



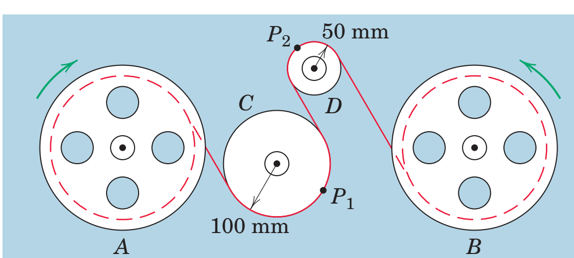
Problem 2/112

2/113 The preliminary design for a "small" space station to orbit the earth in a circular path consists of a ring (torus) with a circular cross section as shown. The living space within the torus is shown in section A, where the "ground level" is 20 ft from the center of the section. Calculate the angular speed N in revolutions per minute required to simulate standard gravity at the surface of the earth (32.17 ft/sec^2). Recall that you would be unaware of a gravitational field if you were in a nonrotating spacecraft in a circular orbit around the earth.



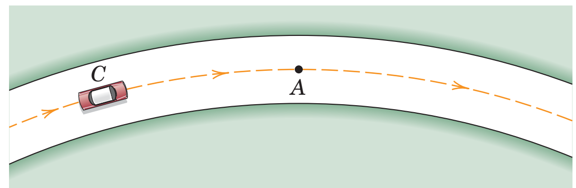
Problem 2/113

2/114 Magnetic tape is being transferred from reel A to reel B and passes around idler pulleys C and D. At a certain instant, point P_1 on the tape is in contact with pulley C and point P_2 is in contact with pulley D. If the normal component of acceleration of P_1 is 40 m/s^2 and the tangential component of acceleration of P_2 is 30 m/s^2 at this instant, compute the corresponding speed v of the tape, the magnitude of the total acceleration of P_1, and the magnitude of the total acceleration of P_2.



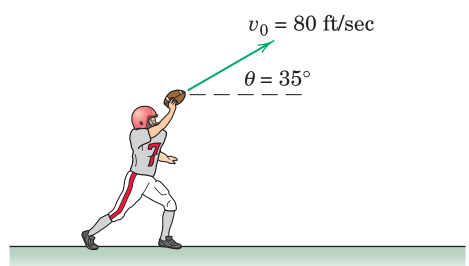
Problem 2/114

2/115 The car C increases its speed at the constant rate of 1.5 m/s^2 as it rounds the curve shown. If the magnitude of the total acceleration of the car is 2.5 m/s^2 at the point A where the radius of curvature is 200 m, compute the speed v of the car at this point.



Problem 2/115

2/116 A football player releases a ball with the initial conditions shown in the figure. Determine the radius of curvature of the trajectory (a) just after release and (b) at the apex. For each case, compute the time rate of change of the speed.



Problem 2/116

2/117 For the football of the previous problem, determine the radius of curvature ρ of the path and the time rate of change $\dot{v}$ of the speed at times $t = 1$ sec and $t = 2$ sec, where $t = 0$ is the time of release from the quarterback's hand.

2/118 A particle moving in the x-y plane has a position vector given by $\mathbf{r} = \frac{3}{2}t^2\mathbf{i} + \frac{2}{3}t^3\mathbf{j}$, where $\mathbf{r}$ is in inches and t is in seconds. Calculate the radius of curvature ρ of the path for the position of the particle when $t = 2$ sec. Sketch the velocity $\mathbf{v}$ and the curvature of the path for this particular instant.

Q2/97

$a_n = 0.88 * 9.81$

$= 8.6328 \ m/sec^2$

$a_n = \frac{v^2}{\rho}$

For car A

$v_A^2 = a_n * \rho$

$= 8.632 * 16$

$\Rightarrow v_A = 11.75 \ m/s \longrightarrow$ Ans

maximum velocity travelled by car A

For car B

$v_B^2 = a_n * \rho$

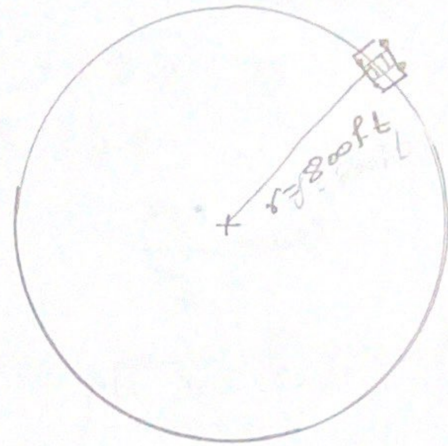
$= 8.623 * 21$

$v_B = 13.46 \ m/sec \longrightarrow$ Ans

maximum velocity travelled by car B.

Q 2/98

$a_t = 10\ ft/sec^2$



$v = 45\ mi/hour$

$= 45 * \frac{5280}{60 \times 60}$

$= 45 * 1.467$

$= 66.015$

$$\overline{a_t} = a_n e_n + a_t e_n$$

$$= \frac{v^2}{r} e_n + \dot{v} e_n$$

$$a_t = \sqrt{\left(\frac{v^2}{r}\right)^2 + (\dot{v})^2} \quad ; \dot{v} = ?$$

$$10 = \sqrt{\left(\frac{66.015^2}{800}\right)^2 + \dot{v}^2}$$

$$\Rightarrow \dot{v} = \pm 8.376 \longrightarrow \text{Ans.}$$

rate change in velocity.

Q 2/100

$$\rho = 98 + 2 = 100 \ m$$

$$\bar{a}_t = a_n e_n + a_t e_t$$

$$= \frac{v^2}{\rho} e_n + v^{\circ} e_t$$; since constant speed

$\Rightarrow$ rate change in velocity equal to zero

$$\Rightarrow a = \sqrt{\left(\frac{v^2}{\rho}\right)^2}$$

$$0.8 * 9.81 = \sqrt{\frac{v^4}{100}}$$

$$\Rightarrow v = 71.3 \ m/sc \longrightarrow \text{Ans.}$$