The Chain Rule قاعدة السلسلة

If y is a differentiable function of x and is a differentiable function of t, then y is differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Ex, / If $y = x^2 + 2x + 1$

$x = 3t^2 + 1$ find $\frac{dy}{dx}$?

method 1 / $y = (3t^2+1)^2 + 2(3t^2+1) + 1$

$= 9t^4 + 12t^2 + 4$

$$\frac{dy}{dx} = 36t^3 + 24t$$

method 2 / by chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = 2x + 2 , \quad \frac{dx}{dt} = 6t$$

$$\frac{dy}{dt} = (2x+2)(6t)$$

$$= [2(3t^2+1)+2](6t)$$

$$= (6t^2+2)(6t)$$

$$= 36t^3 - 24t$$

Ex 2/ find y' for $y = \sin(x^2+6)$

by using chain Rule

let $y = \sin u \rightarrow u = x^2+6$

use chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\frac{dy}{du} = \cos u, \quad \frac{du}{dx} = 2x$$

$$\therefore \frac{dy}{dx} = \cos u \cdot 2x$$

$$= \cos(x^2+6) \cdot 2x$$

$$= 2x \cos(x^2+6)$$

Repeated use for chain Rule:

If $y = f(u)$, $u = g(u)$, and $v = h(x)$

Then $\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

Ex / If $y = \sin(1 + \tan 2x)$ find $\frac{dy}{dx}$

let $y = f(u) = \sin u$, $u = g(v) = 1 + \tan v$

$v = h(x) = 2x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \cos u \cdot \sec^2 v \cdot 2$$

$$= \cos(1 + \tan v) \cdot \sec^2 v \cdot 2$$

$$\frac{dy}{dx} = \cos(1 + \tan 2x) \cdot \sec^2 2x \cdot 2$$

$$= 2\cos(1 + \tan 2x) \cdot \sec^2 2x$$

Ex / If $y = \cos^2 3x$, find $\frac{dy}{dx}$ using chain Rule

Sol:-

$$y = (\cos 3x)^2$$

$$y = f(u) = u^2$$

$$u = \cos v \quad , \quad v = 3x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= 2u \cdot (-\sin v) \cdot 3$$

$$= 2\cos 3x \, (-\sin 3x) \cdot 3$$

$$= -6 \sin 3x \cos 3x$$

$$= -3 \sin 6x$$