Derivatives

Derivatives are the functions we use to mesure the rate at which things change. It can be difined as a limiting values of average changes.

The average rate of change can be represented in some forms as: Speed by time (km/hr) (percent per year), (cm/month), ... etc.

* for any function $f(x)$

$$\text{average rate of change} = \frac{\Delta y}{\Delta x}$$

= Slop of Secant line

Slop of tangent line at p = derivative of $f(x)$ at p = limt of Secant slop as Q approaches p.

Derivative of $f(x)$ = lim [Secant Slop]

$$\text{Secant Slop} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\therefore \text{derivative of } f(x) = \lim \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The Symbols of the derivative of $f(x)$ are:

1- $\frac{dy}{dx}$, 2- y' , 3- $f'(x)$, 4- $\frac{d}{dx}$

5- $D_x y$

* for Second derivative $\Rightarrow$ $\frac{d^2y}{dx^2}$, y'' , $f''(x)$

---etc.

Ex/ IF $f(x) = x^2 \rightarrow$ find $\frac{dy}{dx}$ by using the definition of the derivative.

Sol:- $f(x) = x^2$

$$f(x+\Delta x) = (\Delta x + x)^2$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(\Delta x + x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x + x)^2 - (x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 - 2x\Delta x + x^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x$$

$$= 2x + 0$$

$$= 2x$$

Ex2/ If $y = 5x^3 - 8x^2 - 3x + 4$ find $\frac{dy}{dx}$

by using the definction of the drivative

$$f(x) = 5x^3 - 8x^2 - 3x + 4$$

$$f(x+\Delta x) = 5(x+\Delta x)^3 - 8(x+\Delta x)^2 - 3(x+\Delta x) + 4$$

$$\frac{dy}{dx} = \lim \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5(\Delta x + x)^3 - 8(\Delta x + x)^2 - 3(\Delta x + x) + 4 - 5x^3 - 8x^2 - 3x + 4}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5(x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) - 8(x^2 - 2x\Delta x + \Delta x^2)}{\Delta x}$$

$$- 3x - 3\Delta x + 4 - 5x^3 - 8x^2 + 3x - 4$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{5x^3} + 15x\Delta x^2 + 15x\Delta x^2 + 5\Delta x^3 + \cancel{8x^2} + 16x\Delta x + 8\Delta x^2}{\Delta x}$$

$$- \cancel{3x} - 3\Delta x - \cancel{5x^3} - \cancel{8x^2} + \cancel{3x}$$

$$= \lim_{\Delta x \to 0} \frac{15x^2\Delta x + 15\Delta x^2 x + 5\Delta x^3 + 16x\Delta x + 8\Delta x^2 - 3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}\,[15x^2 + 15x\Delta x + 5\Delta x^2 + 16x + 8\Delta x - 3]}{\cancel{\Delta x}}$$

$$= 15x^2 + 15x(0) + 5(0)^2 + 16x + 8(0) - 3$$

$$= 15x^2 + 16x - 3.$$

Theorem ~~If~~ $f(x)$ has derivative at $x = c$, then $f(x)$ is continuous at $x = c$

Note / the function has derivative at a point if and only if the right-hand and left hand derivative are equal.

EX / If the function $y = |x|$ has derivative at $x = 0$? why?

Sol:- $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

at $x=0 \Rightarrow f'(x) = \lim_{\Delta x \to 0} \frac{|x+\Delta x| - |x|}{\Delta x}$

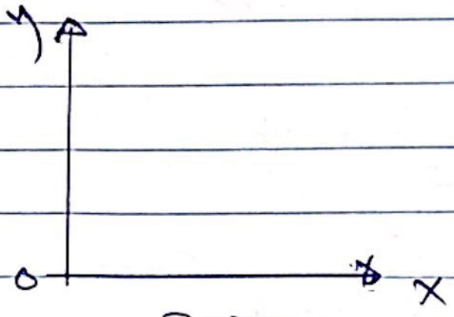
$= \lim_{\Delta x \to 0} \frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$

If $\Delta x > 0 \Rightarrow f'(x) = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1 = R$

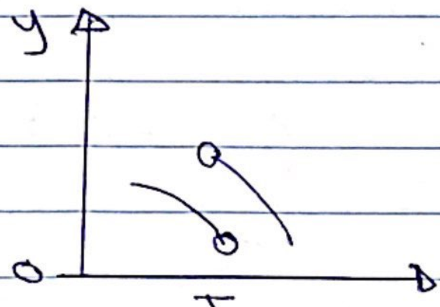
If $\Delta x < 0 \Rightarrow f'(x) = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1 = L$

$1 = -1 \Rightarrow y = |x|$ has no derivative at $x=0$
because $R \neq L \Rightarrow 1 \neq -1$

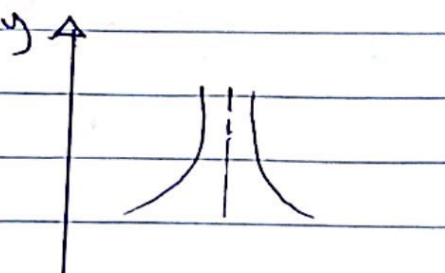
Note/ the cases in which the function has no derivative :-



Corner



Jump



vertical tangent

Ex/ Find an equation for tangent to the curve $y = x + \frac{1}{x}$ at $x = 2$

Sol:-

$$\text{at } x = 2 \Rightarrow y = 2 + \frac{1}{2} = \frac{5}{2}$$

$$P\left(2, \frac{5}{2}\right)$$

$$y = x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 + \frac{-1}{x^2} = 1 - \frac{1}{x^2}$$

$$\text{at } x = 2 \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{(2)^2} = 1 - \frac{1}{4} = \frac{3}{4} = m$$

So we found the slop and point now we can find the equation :-

$$y - y_1 = m(x - x_1)$$

$$y - \frac{5}{2} = \frac{3}{4}(x - 2)$$

$$y - \frac{5}{2} = \frac{3}{4}x - \frac{6}{4}$$

$$y = \frac{3}{4}x - \frac{6}{4} + \frac{5}{2}$$

$$y = \frac{3}{4} + 1$$

Ex/ find the point on the curve $y = x^3 + x^2 - 1$ Where the tangent is parallral to the x-axis

Sol:- $y = x^3 + x^2 - 1$

$$\frac{dy}{dx} = 3x^2 + 2x$$

$$3x^2 + 2x = 0$$
$$x(3x + 2) = 0$$

either $x = 0$

or $3x + 2 = 0 \Rightarrow x = -\frac{2}{3}$

at $x = 0 \Rightarrow y = 0^3 + 0^2 - 1 = -1$ $P_1(0, -1)$

at $x = -\frac{2}{3} \Rightarrow y = 3\left(-\frac{2}{3}\right)^3 + \left(-\frac{2}{3}\right)^2 - 1$

$$= \frac{-23}{27}$$

$$P_2\left(\frac{-2}{3}, \frac{-23}{27}\right)$$