These equations agree with Eqs. (2-44a) and (2-44b) of Section 2.7.

Also, for **pure shear** we substitute

$$\sigma_x = \sigma_y = 0 \qquad \epsilon_x = \epsilon_y = 0$$

into Eqs. (7-50) and (7-51) and obtain

$$u = \frac{\tau_{xy}^2}{2G} \qquad u = \frac{G\gamma_{xy}^2}{2} \tag{g,h}$$

These equations agree with Eqs. (3-55a) and (3-55b) of Section 3.9.

7.6 TRIAXIAL STRESS

An element of material subjected to normal stresses σ_x, σ_y, and σ_z acting in three mutually perpendicular directions is said to be in a state of **triaxial stress** (Fig. 7-27a). Since there are no shear stresses on the x, y, and z faces, the stresses σ_x, σ_y, and σ_z are the *principal stresses* in the material.

If an inclined plane parallel to the z axis is cut through the element (Fig. 7-27b), the only stresses on the inclined face are the normal stress σ and shear stress τ, both of which act parallel to the xy plane. These stresses are analogous to the stresses σ_{x_1} and $\tau_{x_1y_1}$ encountered in our earlier discussions of plane stress (see, for instance, Fig. 7-2a). Because the stresses σ and τ (Fig. 7-27b) are found from equations of force equilibrium in the xy plane, they are independent of the normal stress σ_z. Therefore, we can use the transformation equations of plane stress, as well as Mohr's circle for plane stress, when determining the stresses σ and τ in triaxial stress. The same general conclusion holds for the normal and shear stresses acting on inclined planes cut through the element parallel to the x and y axes.

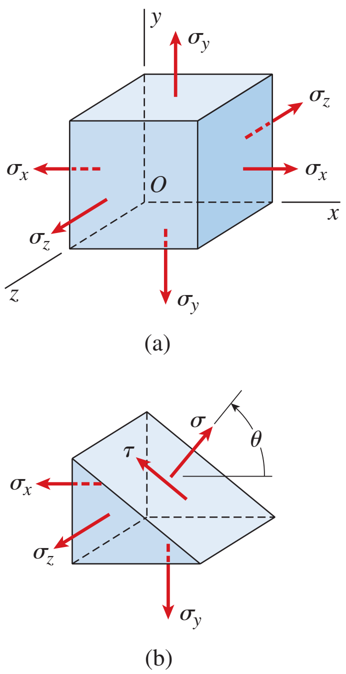


FIG. 7-27 Element in triaxial stress

Maximum Shear Stresses

From our previous discussions of plane stress, we know that the maximum shear stresses occur on planes oriented at 45° to the principal planes. Therefore, for a material in triaxial stress (Fig. 7-27a), the maximum shear stresses occur on elements oriented at angles of 45° to the x, y, and z axes. For example, consider an element obtained by a 45° rotation about the z axis. The maximum positive and negative shear stresses acting on this element are

$$(\tau_{\max})_z = \pm \frac{\sigma_x - \sigma_y}{2} \tag{7-52a}$$

Similarly, by rotating about the x and y axes through angles of 45°, we obtain the following maximum shear stresses:

$$(\tau_{max})_x = \pm \frac{\sigma_y - \sigma_z}{2} \qquad (\tau_{max})_y = \pm \frac{\sigma_x - \sigma_z}{2} \qquad \text{(7-52b,c)}$$

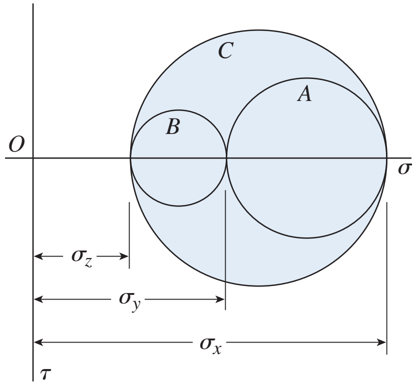


FIG. 7-28 Mohr's circles for an element in triaxial stress

The absolute maximum shear stress is the numerically largest of the stresses determined from Eqs. (7-52a, b, and c). It is equal to one-half the difference between the algebraically largest and algebraically smallest of the three principal stresses.

The stresses acting on elements oriented at various angles to the x, y, and z axes can be visualized with the aid of **Mohr's circles**. For elements oriented by rotations about the z axis, the corresponding circle is labeled A in Fig. 7-28. Note that this circle is drawn for the case in which $\sigma_x > \sigma_y$ and both σ_x and σ_y are tensile stresses.

In a similar manner, we can construct circles B and C for elements oriented by rotations about the x and y axes, respectively. The radii of the circles represent the maximum shear stresses given by Eqs. (7-52a, b, and c), and the absolute maximum shear stress is equal to the radius of the largest circle. The normal stresses acting on the planes of maximum shear stresses have magnitudes given by the abscissas of the centers of the respective circles.

In the preceding discussion of triaxial stress we only considered stresses acting on planes obtained by rotating about the x, y, and z axes. Thus, every plane we considered is parallel to one of the axes. For instance, the inclined plane of Fig. 7-27b is parallel to the z axis, and its normal is parallel to the xy plane. Of course, we can also cut through the element in **skew directions**, so that the resulting inclined planes are skew to all three coordinate axes. The normal and shear stresses acting on such planes can be obtained by a more complicated three-dimensional analysis. However, the normal stresses acting on skew planes are intermediate in value between the algebraically maximum and minimum principal stresses, and the shear stresses on those planes are smaller (in absolute value) than the absolute maximum shear stress obtained from Eqs. (7-52a, b, and c).

Hooke's Law for Triaxial Stress

If the material follows Hooke's law, we can obtain the relationships between the normal stresses and normal strains by using the same procedure as for plane stress (see Section 7.5). The strains produced by the stresses σ_x, σ_y, and σ_z acting independently are superimposed to obtain the resultant strains. Thus, we readily arrive at the following equations for the **strains in triaxial stress**:

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \qquad \text{(7-53a)}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x) \qquad \text{(7-53b)}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) \qquad \text{(7-53c)}$$

In these equations, the standard sign conventions are used; that is, tensile stress σ and extensional strain ϵ are positive.

The preceding equations can be solved simultaneously for the **stresses in terms of the strains**:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z)\right] \tag{7-54a}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_y + \nu(\epsilon_z + \epsilon_x)\right] \tag{7-54b}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_z + \nu(\epsilon_x + \epsilon_y)\right] \tag{7-54c}$$

Equations (7-53) and (7-54) represent **Hooke's law for triaxial stress**.

In the special case of **biaxial stress** (Fig. 7-11b), we can obtain the equations of Hooke's law by substituting $\sigma_z = 0$ into the preceding equations. The resulting equations reduce to Eqs. (7-39) and (7-40) of Section 7.5.

Unit Volume Change

The unit volume change (or *dilatation*) for an element in triaxial stress is obtained in the same manner as for plane stress (see Section 7.5). If the element is subjected to strains ϵ_x, ϵ_y, and ϵ_z, we may use Eq. (7-46) for the unit volume change:

$$e = \epsilon_x + \epsilon_y + \epsilon_z \tag{7-55}$$

This equation is valid for any material provided the strains are small.

If Hooke's law holds for the material, we can substitute for the strains ϵ_x, ϵ_y, and ϵ_z from Eqs. (7-53a, b, and c) and obtain

$$e = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{7-56}$$

Equations (7-55) and (7-56) give the unit volume change in triaxial stress in terms of the strains and stresses, respectively.

Strain-Energy Density

The strain-energy density for an element in triaxial stress is obtained by the same method used for plane stress. When stresses σ_x and σ_y act alone (biaxial stress), the strain-energy density (from Eq. 7-49 with the shear term discarded) is

$$u = \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y)$$

When the element is in triaxial stress and subjected to stresses σ_x, σ_y, and σ_z, the expression for strain-energy density becomes

$$u = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z) \tag{7-57a}$$

Substituting for the strains from Eqs. (7-53a, b, and c), we obtain the strain-energy density in terms of the stresses:

$$u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z) \tag{7-57b}$$

In a similar manner, but using Eqs. (7-54a, b, and c), we can express the strain-energy density in terms of the strains:

$$u = \frac{E}{2(1 + \nu)(1 - 2\nu)}[(1 - \nu)(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + 2\nu(\epsilon_x \epsilon_y + \epsilon_x \epsilon_z + \epsilon_y \epsilon_z)] \tag{7-57c}$$

When calculating from these expressions, we must be sure to substitute the stresses and strains with their proper algebraic signs.

Spherical Stress

A special type of triaxial stress, called **spherical stress**, occurs whenever all three normal stresses are equal (Fig. 7-29):

$$\sigma_x = \sigma_y = \sigma_z = \sigma_0 \tag{7-58}$$

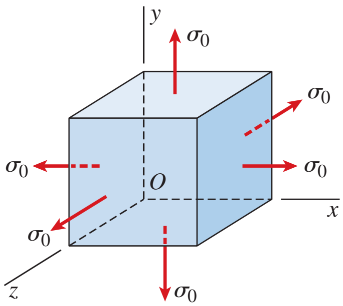


FIG. 7-29 Element in spherical stress

Under these stress conditions, *any* plane cut through the element will be subjected to the same normal stress σ_0 and will be free of shear stress. Thus, we have equal normal stresses in every direction and no shear stresses anywhere in the material. Every plane is a principal plane, and the three Mohr's circles shown in Fig. 7-28 reduce to a single point.

The normal strains in spherical stress are also the same in all directions, provided the material is homogeneous and isotropic. If Hooke's law applies, the normal strains are

$$\epsilon_0 = \frac{\sigma_0}{E}(1 - 2\nu) \tag{7-59}$$

as obtained from Eqs. (7-53a, b, and c).

Since there are no shear strains, an element in the shape of a cube changes in size but remains a cube. In general, any body subjected to spherical stress will maintain its relative proportions but will expand or contract in volume depending upon whether σ_0 is tensile or compressive.

The expression for the unit volume change can be obtained from Eq. (7-55) by substituting for the strains from Eq. (7-59). The result is

$$e = 3\epsilon_0 = \frac{3\sigma_0(1 - 2\nu)}{E} \tag{7-60}$$

Equation (7-60) is usually expressed in more compact form by introducing a new quantity K called the **volume modulus of elasticity**, or **bulk modulus of elasticity**, which is defined as follows:

$$K = \frac{E}{3(1 - 2\nu)} \tag{7-61}$$

With this notation, the expression for the unit volume change becomes

$$e = \frac{\sigma_0}{K} \tag{7-62}$$

and the volume modulus is

$$K = \frac{\sigma_0}{e} \tag{7-63}$$

Thus, the volume modulus can be defined as the ratio of the spherical stress to the volumetric strain, which is analogous to the definition of the modulus E in uniaxial stress. Note that the preceding formulas for e and K are based upon the assumptions that *the strains are small and Hooke's law holds for the material*.

From Eq. (7-61) for K, we see that if Poisson's ratio ν equals 1/3, the moduli K and E are numerically equal. If $\nu = 0$, then K has the value $E/3$, and if $\nu = 0.5$, K becomes infinite, which corresponds to a rigid material having no change in volume (that is, the material is incompressible).

The preceding formulas for spherical stress were derived for an element subjected to uniform tension in all directions, but of course the formulas also apply to an element in uniform compression. In the case of uniform compression, the stresses and strains have negative signs. Uniform compression occurs when the material is subjected to uniform pressure in all directions; for example, an object submerged in water or rock deep within the earth. This state of stress is often called **hydrostatic stress**.

Although uniform compression is relatively common, a state of uniform tension is difficult to achieve. It can be realized by suddenly and uniformly heating the outer surface of a solid metal sphere, so that the outer layers are at a higher temperature than the interior. The tendency of the outer layers to expand produces uniform tension in all directions at the center of the sphere.

7.7 PLANE STRAIN

The strains at a point in a loaded structure vary according to the orientation of the axes, in a manner similar to that for stresses. In this section we will derive the transformation equations that relate the strains in inclined directions to the strains in the reference directions. These transformation equations are widely used in laboratory and field investigations involving measurements of strains.

Strains are customarily measured by *strain gages*; for example, gages are placed in aircraft to measure structural behavior during flight, and gages are placed in buildings to measure the effects of earthquakes. Since each gage measures the strain in one particular direction, it is usually necessary to calculate the strains in other directions by means of the transformation equations.

Plane Strain Versus Plane Stress

Let us begin by explaining what is meant by plane strain and how it relates to plane stress. Consider a small element of material having sides of lengths a, b, and c in the x, y, and z directions, respectively (Fig. 7-30a). If the only deformations are those in the xy plane, then three strain components may exist—the normal strain ϵ_x in the x direction (Fig. 7-30b), the normal strain ϵ_y in the y direction (Fig. 7-30c), and the shear strain γ_{xy} (Fig. 7-30d). An element of material subjected to these strains (and *only* these strains) is said to be in a state of **plane strain**.

It follows that an element in plane strain has no normal strain ϵ_z in the z direction and no shear strains γ_{xz} and γ_{yz} in the xz and yz planes, respectively. Thus, plane strain is defined by the following conditions:

$$\epsilon_z = 0 \qquad \gamma_{xz} = 0 \qquad \gamma_{yz} = 0 \tag{7-64a,b,c}$$

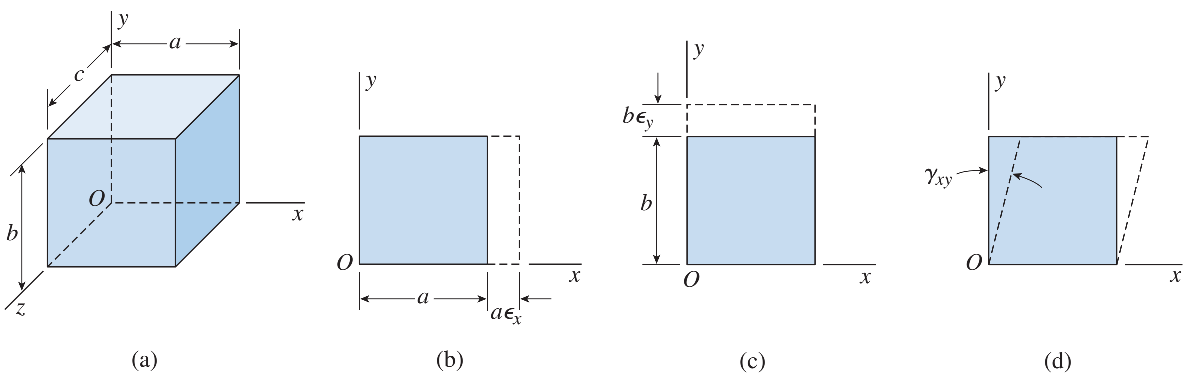
The remaining strains (ϵ_x, ϵ_y, and γ_{xy}) may have nonzero values.

From the preceding definition, we see that plane strain occurs when the front and rear faces of an element of material (Fig. 7-30a) are fully restrained against displacement in the z direction—an idealized condition that is seldom reached in actual structures. However, this does not mean that the transformation equations of plane strain are not useful. It turns out that they are extremely useful because they also apply to the strains in plane stress, as explained in the following paragraphs.

The definition of plane strain (Eqs. 7-64a, b, and c) is analogous to that for plane stress. In plane stress, the following stresses must be zero:

$$\sigma_z = 0 \qquad \tau_{xz} = 0 \qquad \tau_{yz} = 0 \tag{7-65a,b,c}$$

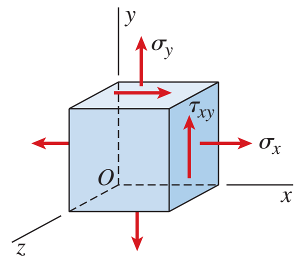
FIG. 7-30 Strain components ϵ_x, ϵ_y, and γ_{xy} in the xy plane (plane strain)

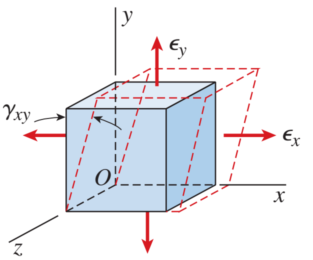


whereas the remaining stresses (σ_x, σ_y, and τ_{xy}) may have nonzero values. A comparison of the stresses and strains in plane stress and plane strain is given in Fig. 7-31.

It should not be inferred from the similarities in the definitions of plane stress and plane strain that both occur simultaneously. In general, an element in plane stress will undergo a strain in the z direction (Fig. 7-31); hence, it is *not* in plane strain. Also, an element in plane strain usually will have stresses σ_z acting on it because of the requirement that $\epsilon_z = 0$; therefore, it is *not* in plane stress. Thus, under ordinary conditions plane stress and plane strain do not occur simultaneously.

An exception occurs when an element in plane stress is subjected to equal and opposite normal stresses (that is, when $\sigma_x = -\sigma_y$) and Hooke's law holds for the material. In this special case, there is no normal strain in the z direction, as shown by Eq. (7-34c), and therefore the element is in a state of plane strain as well as plane stress. Another special case, albeit a hypothetical one, is when a material has Poisson's ratio equal to zero ($\nu = 0$); then every plane stress element is also in plane strain because $\epsilon_z = 0$ (Eq. 7-34c).*





	Plane stress	**Plane strain**
Stresses	$\sigma_z = 0$ $\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, and τ_{xy} may have nonzero values	$\tau_{xz} = 0$ $\tau_{yz} = 0$ σ_x, σ_y, σ_z, and τ_{xy} may have nonzero values
Strains	$\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ϵ_x, ϵ_y, ϵ_z, and γ_{xy} may have nonzero values	$\epsilon_z = 0$ $\gamma_{xz} = 0$ $\gamma_{yz} = 0$ ϵ_x, ϵ_y, and γ_{xy} may have nonzero values

FIG. 7-31 Comparison of plane stress and plane strain

*In the discussions of this chapter we are omitting the effects of temperature changes and prestrains, both of which produce additional deformations that may alter some of our conclusions.

Application of the Transformation Equations

The stress-transformation equations derived for plane stress in the xy plane (Eqs. 7-4a and 7-4b) are valid even when a normal stress σ_z is present. The explanation lies in the fact that the stress σ_z does not enter the equations of equilibrium used in deriving Eqs. (7-4a) and (7-4b). Therefore, *the transformation equations for plane stress can also be used for the stresses in plane strain*.

An analogous situation exists for plane strain. Although we will derive the strain-transformation equations for the case of plane strain in the xy plane, the equations are valid even when a strain ϵ_z exists. The reason is simple enough—the strain ϵ_z does not affect the geometric relationships used in the derivations. Therefore, *the transformation equations for plane strain can also be used for the strains in plane stress.*

Finally, we should recall that the transformation equations for plane stress were derived solely from equilibrium and therefore are valid for any material, whether linearly elastic or not. The same conclusion applies to the transformation equations for plane strain—since they are derived solely from geometry, *they are independent of the material properties*.

Transformation Equations for Plane Strain

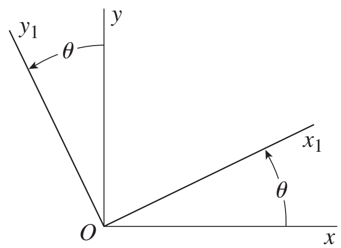


FIG. 7-32 Axes x_1 and y_1 rotated through an angle θ from the xy axes

In the derivation of the transformation equations for plane strain, we will use the coordinate axes shown in Fig. 7-32. We will assume that the normal strains ϵ_x and ϵ_y and the shear strain γ_{xy} associated with the xy axes are known (Fig. 7-30). The objectives of our analysis are to determine the normal strain ϵ_{x_1} and the shear strain $\gamma_{x_1 y_1}$ associated with the $x_1 y_1$ axes, which are rotated counterclockwise through an angle θ from the xy axes. (It is not necessary to derive a separate equation for the normal strain ϵ_{y_1} because it can be obtained from the equation for ϵ_{x_1} by substituting $\theta + 90°$ for θ.)

Normal strain ϵ_{x_1}. To determine the normal strain ϵ_{x_1} in the x_1 direction, we consider a small element of material selected so that the x_1 axis is along a diagonal of the z face of the element and the x and y axes are along the sides of the element (Fig. 7-33a). The figure shows a two-dimensional view of the element, with the z axis toward the viewer. Of course, the element is actually three dimensional, as in Fig. 7-30a, with a dimension in the z direction.

Consider first the strain ϵ_x in the x direction (Fig. 7-33a). This strain produces an elongation in the x direction equal to $\epsilon_x dx$, where dx is the length of the corresponding side of the element. As a result of this elongation, the diagonal of the element increases in length by an amount

$$\epsilon_x \, dx \cos \theta \qquad \text{(a)}$$

as shown in Fig. 7-33a.

Next, consider the strain ϵ_y in the y direction (Fig. 7-33b). This strain produces an elongation in the y direction equal to $\epsilon_y dy$, where dy is the length of the side of the element parallel to the y axis. As a result of this elongation, the diagonal of the element increases in length by an amount

$$\epsilon_y \, dy \sin \theta \tag{b}$$

which is shown in Fig. 7-33b.

Finally, consider the shear strain γ_{xy} in the xy plane (Fig. 7-33c). This strain produces a distortion of the element such that the angle at the lower left corner of the element decreases by an amount equal to the shear strain. Consequently, the upper face of the element moves to the right (with respect to the lower face) by an amount $\gamma_{xy} dy$. This deformation results in an increase in the length of the diagonal equal to

$$\gamma_{xy} \, dy \cos \theta \tag{c}$$

as shown in Fig. 7-33c.

The total increase Δd in the length of the diagonal is the sum of the preceding three expressions; thus,

$$\Delta d = \epsilon_x \, dx \cos \theta + \epsilon_y \, dy \sin \theta + \gamma_{xy} \, dy \cos \theta \tag{d}$$

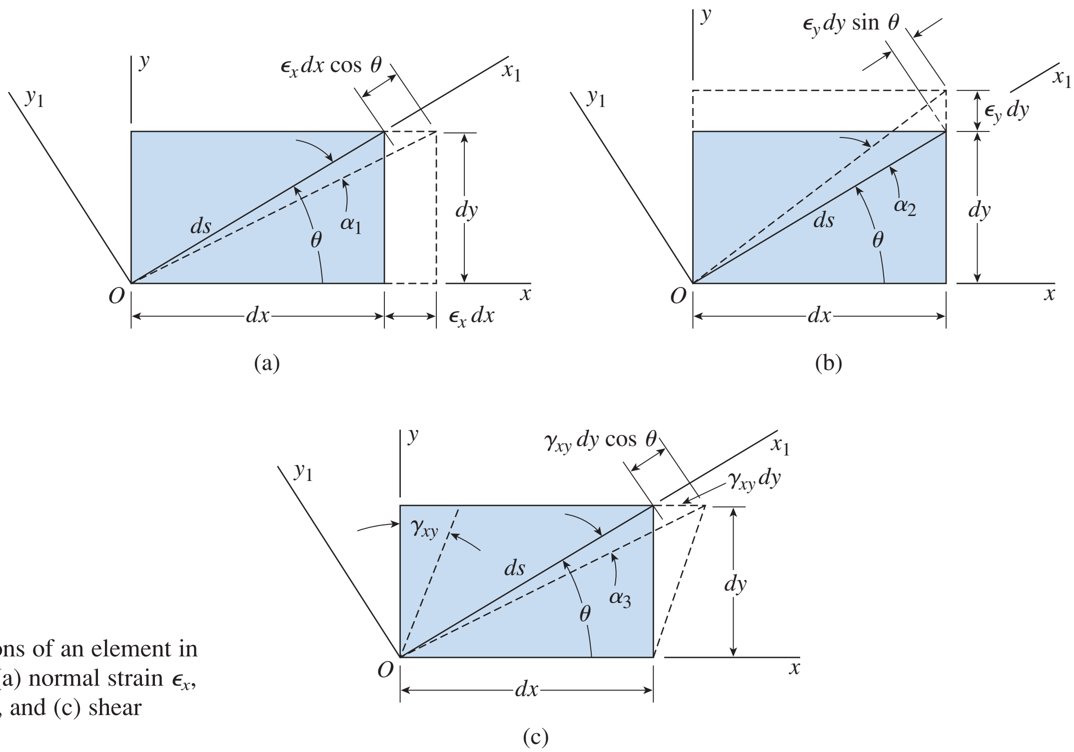


FIG. 7-33 Deformations of an element in plane strain due to (a) normal strain ϵ_x, (b) normal strain ϵ_y, and (c) shear strain γ_{xy}

The normal strain ϵ_{x_1} in the x_1 direction is equal to this increase in length divided by the initial length ds of the diagonal:

$$\epsilon_{x_1} = \frac{\Delta d}{ds} = \epsilon_x \frac{dx}{ds}\cos\theta + \epsilon_y \frac{dy}{ds}\sin\theta + \gamma_{xy}\frac{dy}{ds}\cos\theta \qquad \text{(e)}$$

Observing that $dx/ds = \cos\theta$ and $dy/ds = \sin\theta$, we obtain the following equation for the **normal strain**:

$$\epsilon_{x_1} = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \qquad \text{(7-66)}$$

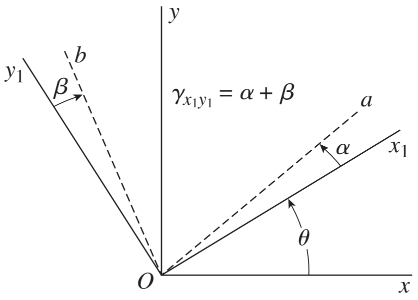


FIG. 7-34 Shear strain $\gamma_{x_1y_1}$ associated with the x_1y_1 axes

Thus, we have obtained an expression for the normal strain in the x_1 direction in terms of the strains ϵ_x, ϵ_y, and γ_{xy} associated with the xy axes.

As mentioned previously, the normal strain ϵ_{y_1} in the y_1 direction is obtained from the preceding equation by substituting $\theta + 90°$ for θ.

Shear strain $\gamma_{x_1y_1}$. Now we turn to the shear strain $\gamma_{x_1y_1}$ associated with the x_1y_1 axes. This strain is equal to the decrease in angle between lines in the material that were initially along the x_1 and y_1 axes. To clarify this idea, consider Fig. 7-34, which shows both the xy and x_1y_1 axes, with the angle θ between them. Let line Oa represent a line in the material that *initially* was along the x_1 axis (that is, along the diagonal of the element in Fig. 7-33). The deformations caused by the strains ϵ_x, ϵ_y, and γ_{xy} (Fig. 7-33) cause line Oa to rotate through a counterclockwise angle α from the x_1 axis to the position shown in Fig. 7-34. Similarly, line Ob was originally along the y_1 axis, but because of the deformations it rotates through a clockwise angle β. The shear strain $\gamma_{x_1y_1}$ is the decrease in angle between the two lines that originally were at right angles; therefore,

$$\gamma_{x_1y_1} = \alpha + \beta \qquad \text{(7-67)}$$

Thus, in order to find the shear strain $\gamma_{x_1y_1}$, we must determine the angles α and β.

The angle α can be found from the deformations pictured in Fig. 7-33 as follows. The strain ϵ_x (Fig. 7-33a) produces a clockwise rotation of the diagonal of the element. Let us denote this angle of rotation as α_1. The angle α_1 is equal to the distance $\epsilon_x\, dx \sin\theta$ divided by the length ds of the diagonal:

$$\alpha_1 = \epsilon_x \frac{dx}{ds}\sin\theta \qquad \text{(f)}$$

Similarly, the strain ϵ_y produces a counterclockwise rotation of the diagonal through an angle α_2 (Fig. 7-33b). This angle is equal to the distance $\epsilon_y\, dy \cos\theta$ divided by ds:

$$\alpha_2 = \epsilon_y \frac{dy}{ds} \cos\theta \tag{g}$$

Finally, the strain γ_{xy} produces a clockwise rotation through an angle α_3 (Fig. 7-33c) equal to the distance $\gamma_{xy}\, dy \sin\theta$ divided by ds:

$$\alpha_3 = \gamma_{xy} \frac{dy}{ds} \sin\theta \tag{h}$$

Therefore, the resultant counterclockwise rotation of the diagonal (Fig. 7-33), equal to the angle α shown in Fig. 7-34, is

$$\begin{aligned}\alpha &= -\alpha_1 + \alpha_2 - \alpha_3 \\ &= -\epsilon_x \frac{dx}{ds} \sin\theta + \epsilon_y \frac{dy}{ds} \cos\theta - \gamma_{xy} \frac{dy}{ds} \sin\theta \end{aligned} \tag{i}$$

Again observing that $dx/ds = \cos\theta$ and $dy/ds = \sin\theta$, we obtain

$$\alpha = -(\epsilon_x - \epsilon_y) \sin\theta \cos\theta - \gamma_{xy} \sin^2\theta \tag{7-68}$$

The rotation of line Ob (Fig. 7-34), which initially was at 90° to line Oa, can be found by substituting $\theta + 90°$ for θ in the expression for α. The resulting expression is counterclockwise when positive (because α is counterclockwise when positive), hence it is equal to the negative of the angle β (because β is positive when clockwise). Thus,

$$\begin{aligned}\beta &= (\epsilon_x - \epsilon_y) \sin(\theta + 90°) \cos(\theta + 90°) + \gamma_{xy} \sin^2(\theta + 90°) \\ &= -(\epsilon_x - \epsilon_y) \sin\theta \cos\theta + \gamma_{xy} \cos^2\theta \end{aligned} \tag{7-69}$$

Adding α and β gives the shear strain $\gamma_{x_1y_1}$ (see Eq. 7-67):

$$\gamma_{x_1y_1} = -2(\epsilon_x - \epsilon_y) \sin\theta \cos\theta + \gamma_{xy} (\cos^2\theta - \sin^2\theta) \tag{j}$$

To put the equation in a more useful form, we divide each term by 2:

$$\frac{\gamma_{x_1y_1}}{2} = -(\epsilon_x - \epsilon_y) \sin\theta \cos\theta + \frac{\gamma_{xy}}{2} (\cos^2\theta - \sin^2\theta) \tag{7-70}$$

We have now obtained an expression for the **shear strain** $\gamma_{x_1y_1}$ associated with the x_1y_1 axes in terms of the strains ϵ_x, ϵ_y, and γ_{xy} associated with the xy axes.

Transformation equations for plane strain. The equations for plane strain (Eqs. 7-66 and 7-70) can be expressed in terms of the angle 2θ by using the following trigonometric identities:

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \qquad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

Thus, the transformation equations for plane strain become

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \qquad (7\text{-}71a)$$

and

$$\frac{\gamma_{x_1 y_1}}{2} = -\frac{\epsilon_x - \epsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \qquad (7\text{-}71b)$$

These equations are the counterparts of Eqs. (7-4a) and (7-4b) for plane stress.

When comparing the two sets of equations, note that ϵ_{x_1} corresponds to σ_{x_1}, $\gamma_{x_1y_1}/2$ corresponds to $\tau_{x_1y_1}$, ϵ_x corresponds to σ_x, ϵ_y corresponds to σ_y, and $\gamma_{xy}/2$ corresponds to τ_{xy}. The corresponding variables in the two sets of transformation equations are listed in Table 7-1.

TABLE 7-1 CORRESPONDING VARIABLES IN THE TRANSFORMATION EQUATIONS FOR PLANE STRESS (EQS. 7-4a AND b) AND PLANE STRAIN (EQS. 7-71a AND b)

Stresses	Strains
σ_x	ϵ_x
σ_y	ϵ_y
τ_{xy}	$\gamma_{xy}/2$
σ_{x_1}	ϵ_{x_1}
$\tau_{x_1y_1}$	$\gamma_{x_1y_1}/2$

The analogy between the transformation equations for plane stress and those for plane strain shows that all of the observations made in Sections 7.2, 7.3, and 7.4 concerning plane stress, principal stresses, maximum shear stresses, and Mohr's circle have their counterparts in plane strain. For instance, the sum of the normal strains in perpendicular directions is a constant (compare with Eq. 7-6):

$$\epsilon_{x_1} + \epsilon_{y_1} = \epsilon_x + \epsilon_y \qquad (7\text{-}72)$$

This equality can be verified easily by substituting the expressions for ϵ_{x_1} (from Eq. 7-71a) and ϵ_{y_1} (from Eq. 7-71a with θ replaced by $\theta + 90°$).

Principal Strains

Principal strains exist on perpendicular planes with the principal angles θ_p calculated from the following equation (compare with Eq. 7-11):

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} \qquad (7\text{-}73)$$

The principal strains can be calculated from the equation

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad (7\text{-}74)$$

which corresponds to Eq. (7-17) for the principal stresses. The two principal strains (in the xy plane) can be correlated with the two principal directions using the technique described in Section 7.3 for the principal stresses. (This technique is illustrated later in Example 7-7.) Finally, note that in plane strain the third principal strain is $\epsilon_z = 0$. Also, the shear strains are zero on the principal planes.

Maximum Shear Strains

The maximum shear strains in the xy plane are associated with axes at 45° to the directions of the principal strains. The algebraically maximum shear strain (in the xy plane) is given by the following equation (compare with Eq. 7-25):

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad (7\text{-}75)$$

The minimum shear strain has the same magnitude but is negative. In the directions of maximum shear strain, the normal strains are

$$\epsilon_{aver} = \frac{\epsilon_x + \epsilon_y}{2} \qquad (7\text{-}76)$$

which is analogous to Eq. (7-27) for stresses. The maximum out-of-plane shear strains, that is, the shear strains in the xz and yz planes, can be obtained from equations analogous to Eq. (7-75).

An element in plane stress that is oriented to the principal directions of stress (see Fig. 7-13b) has no shear stresses acting on its faces. Therefore, the shear strain $\gamma_{x_1 y_1}$ for this element is zero. It follows that the normal strains in this element are the principal strains. Thus, at a given point in a stressed body, *the principal strains and principal stresses occur in the same directions*.

Mohr's Circle for Plane Strain

Mohr's circle for plane strain is constructed in the same manner as the circle for plane stress, as illustrated in Fig. 7-35. Normal strain ϵ_{x_1} is plotted as the abscissa (positive to the right) and one-half the shear strain ($\gamma_{x_1 y_1}/2$) is plotted as the ordinate (positive downward). The center C of the circle has an abscissa equal to ϵ_{aver} (Eq. 7-76).

Point A, representing the strains associated with the x direction ($\theta = 0$), has coordinates ϵ_x and $\gamma_{xy}/2$. Point B, at the opposite end of a diameter from A, has coordinates ϵ_y and $-\gamma_{xy}/2$, representing the strains associated with a pair of axes rotated through an angle $\theta = 90°$.

The strains associated with axes rotated through an angle θ are given by point D, which is located on the circle by measuring an angle 2θ counterclockwise from radius CA. The principal strains are represented by points P_1 and P_2, and the maximum shear strains by points S_1 and S_2. All of these strains can be determined from the geometry of the circle or from the transformation equations.

Strain Measurements

An electrical-resistance **strain gage** is a device for measuring normal strains on the surface of a stressed object. These gages are quite small, with lengths typically in the range from one-eighth to one-half of an inch. The gages are bonded securely to the surface of the object so that they change in length in proportion to the strains in the object itself.

Each gage consists of a fine metal grid that is stretched or shortened when the object is strained at the point where the gage is attached. The grid is equivalent to a continuous wire that goes back and forth from one end of the grid to the other, thereby effectively increasing its length (Fig. 7-36). The electrical resistance of the wire is altered when it stretches or shortens—then this change in resistance is converted into a measurement of strain. The gages are extremely sensitive and can measure strains as small as 1×10^{-6}.

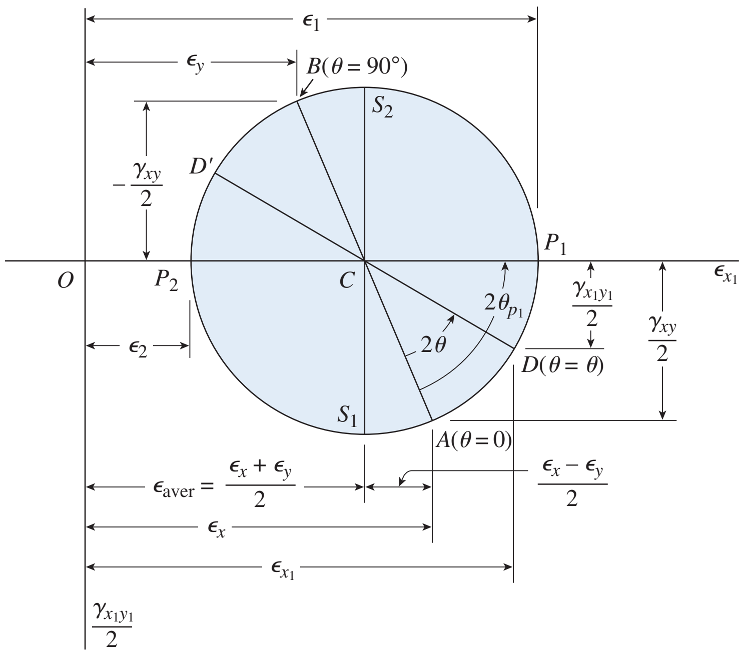


FIG. 7-35 Mohr's circle for plane strain

Since each gage measures the normal strain in only one direction, and since the directions of the principal stresses are usually unknown, it is necessary to use three gages in combination, with each gage measuring the strain in a different direction. From three such measurements, it is possible to calculate the strains in any direction, as illustrated in Example 7-8.

A group of three gages arranged in a particular pattern is called a **strain rosette**. Because the rosette is mounted on the surface of the body, where the material is in plane stress, we can use the transformation equations for plane strain to calculate the strains in various directions. (As explained earlier in this section, the transformation equations for plane strain can also be used for the strains in plane stress.)

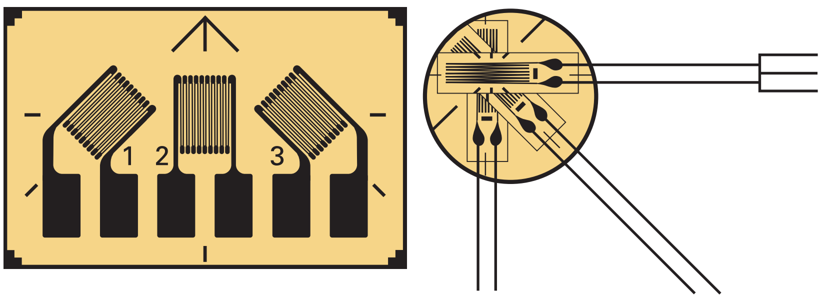
Calculation of Stresses from the Strains

The strain equations presented in this section are derived solely from geometry, as already pointed out. Therefore, the equations apply to any material, whether linear or nonlinear, elastic or inelastic. However, if it is desired to determine the stresses from the strains, the material properties must be taken into account.

If the material follows Hooke's law, we can find the stresses using the appropriate stress-strain equations from either Section 7.5 (for plane stress) or Section 7.6 (for triaxial stress).

As a first example, suppose that the material is in plane stress and that we know the strains ϵ_x, ϵ_y, and γ_{xy}, perhaps from strain-gage measurements. Then we can use the stress-strain equations for plane stress (Eqs. 7-36 and 7-37) to obtain the stresses in the material.

Now consider a second example. Suppose we have determined the three principal strains ϵ_1, ϵ_2, and ϵ_3 for an element of material (if the element is in plane strain, then $\epsilon_3 = 0$). Knowing these strains, we can find the principal stresses using Hooke's law for triaxial stress (see Eqs. 7-54a, b, and c). Once the principal stresses are known, we can find the stresses on inclined planes using the transformation equations for plane stress (see the discussion at the beginning of Section 7.6).



(a) 45° strain gages three-element rosette

(b) Three-element strain-gage rosettes prewired

FIG. 7-36 Three electrical-resistance strain gages arranged as a 45° strain rosette (magnified view).

Example 7-7

An element of material in plane strain undergoes the following strains:

$$\epsilon_x = 340 \times 10^{-6} \qquad \epsilon_y = 110 \times 10^{-6} \qquad \gamma_{xy} = 180 \times 10^{-6}$$

These strains are shown highly exaggerated in Fig. 7-37a, which shows the deformations of an element of unit dimensions. Since the edges of the element have unit lengths, the changes in linear dimensions have the same magnitudes as the normal strains ϵ_x and ϵ_y. The shear strain γ_{xy} is the decrease in angle at the lower-left corner of the element.

Determine the following quantities: (a) the strains for an element oriented at an angle $\theta = 30°$, (b) the principal strains, and (c) the maximum shear strains. (Consider only the in-plane strains, and show all results on sketches of properly oriented elements.)

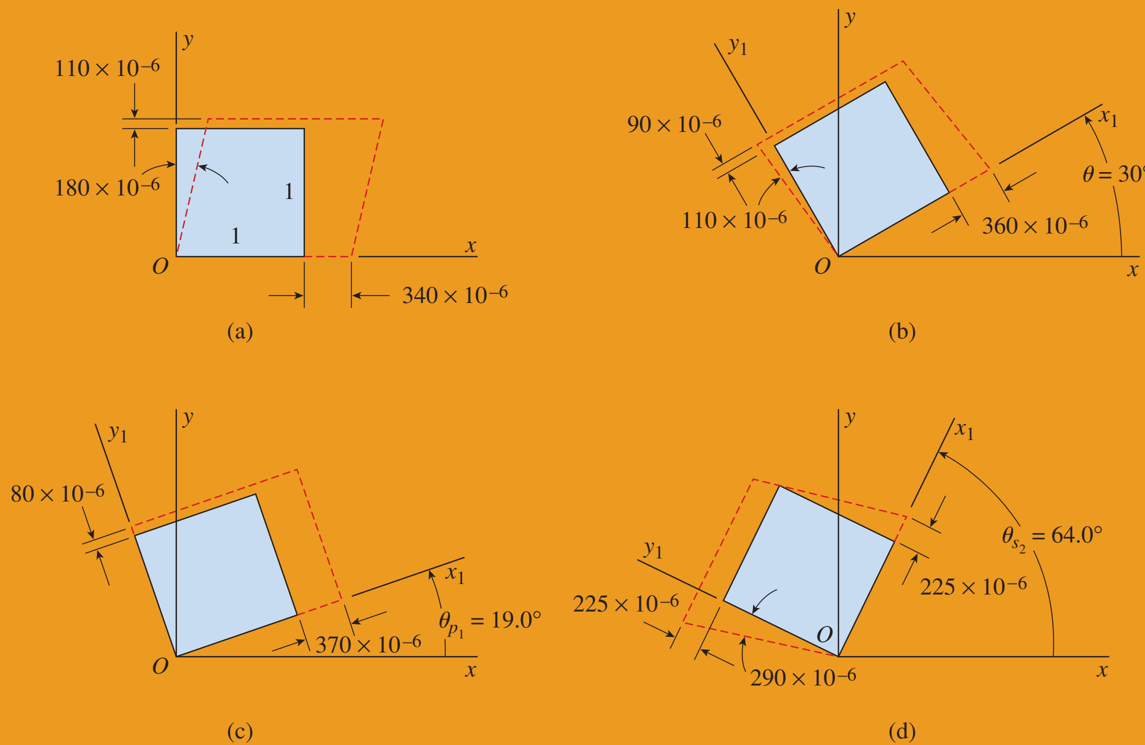


FIG. 7-37 Example 7-7. Element of material in plane strain: (a) element oriented to the x and y axes, (b) element oriented at an angle $\theta = 30°$, (c) principal strains, and (d) maximum shear strains. (*Note:* The edges of the elements have unit lengths.)

continued

Solution

(a) *Element oriented at an angle* $\theta = 30°$. The strains for an element oriented at an angle θ to the x axis can be found from the transformation equations (Eqs. 7-71a and 7-71b). As a preliminary matter, we make the following calculations:

$$\frac{\epsilon_x + \epsilon_y}{2} = \frac{(340 + 110)10^{-6}}{2} = 225 \times 10^{-6}$$

$$\frac{\epsilon_x - \epsilon_y}{2} = \frac{(340 - 110)10^{-6}}{2} = 115 \times 10^{-6}$$

$$\frac{\gamma_{xy}}{2} = 90 \times 10^{-6}$$

Now substituting into Eqs. (7-71a) and (7-71b), we get

$$\begin{aligned}\epsilon_{x_1} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \\ &= (225 \times 10^{-6}) + (115 \times 10^{-6})(\cos 60°) + (90 \times 10^{-6})(\sin 60°) \\ &= 360 \times 10^{-6}\end{aligned}$$

$$\begin{aligned}\frac{\gamma_{x_1y_1}}{2} &= -\frac{\epsilon_x - \epsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \\ &= -(115 \times 10^{-6})(\sin 60°) + (90 \times 10^{-6})(\cos 60°) \\ &= -55 \times 10^{-6}\end{aligned}$$

Therefore, the shear strain is

$$\gamma_{x_1y_1} = -110 \times 10^{-6}$$

The strain ϵ_{y_1} can be obtained from Eq. (7-72), as follows:

$$\epsilon_{y_1} = \epsilon_x + \epsilon_y - \epsilon_{x_1} = (340 + 110 - 360)10^{-6} = 90 \times 10^{-6}$$

The strains ϵ_{x_1}, ϵ_{y_1}, and $\gamma_{x_1y_1}$ are shown in Fig. 7-37b for an element oriented at $\theta = 30°$. Note that the angle at the lower-left corner of the element increases because $\gamma_{x_1y_1}$ is negative.

(b) *Principal strains*. The principal strains are readily determined from Eq. (7-74), as follows:

$$\begin{aligned}\epsilon_{1,2} &= \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 225 \times 10^{-6} \pm \sqrt{(115 \times 10^{-6})^2 + (90 \times 10^{-6})^2} \\ &= 225 \times 10^{-6} \pm 146 \times 10^{-6}\end{aligned}$$

Thus, the principal strains are

$$\epsilon_1 = 370 \times 10^{-6} \qquad \epsilon_2 = 80 \times 10^{-6}$$

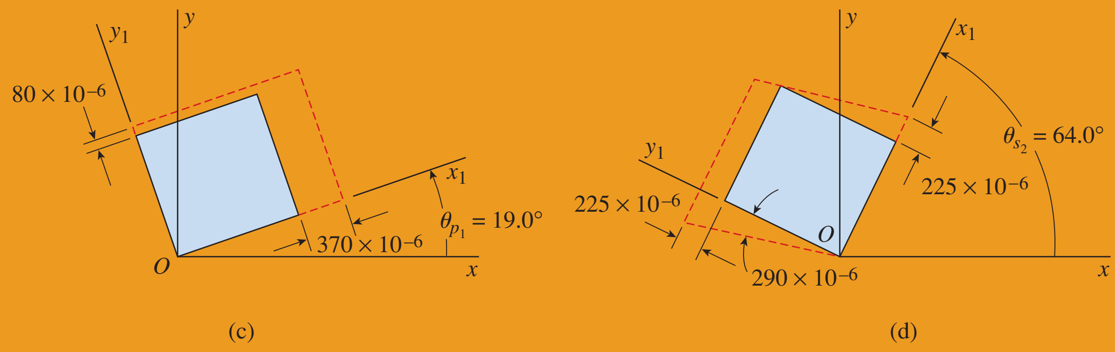


FIG. 7-37c and d (Repeated)

in which ϵ_1 denotes the algebraically larger principal strain and ϵ_2 denotes the algebraically smaller principal strain. (Recall that we are considering only in-plane strains in this example.)

The angles to the principal directions can be obtained from Eq. (7-73):

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = \frac{180}{340 - 110} = 0.7826$$

The values of $2\theta_p$ between 0 and 360° are 38.0° and 218.0°, and therefore the angles to the principal directions are

$$\theta_p = 19.0° \text{ and } 109.0°$$

To determine the value of θ_p associated with each principal strain, we substitute $\theta_p = 19.0°$ into the first transformation equation (Eq. 7-71a) and solve for the strain:

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x + \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$= (225 \times 10^{-6}) + (115 \times 10^{-6})(\cos 38.0°) + (90 \times 10^{-6})(\sin 38.0°)$$

$$= 370 \times 10^{-6}$$

This result shows that the larger principal strain ϵ_1 is at the angle $\theta_{p_1} = 19.0°$. The smaller strain ϵ_2 acts at 90° from that direction ($\theta_{p_2} = 109.0°$). Thus,

$$\epsilon_1 = 370 \times 10^{-6} \quad \text{and} \quad \theta_{p_1} = 19.0°$$

$$\epsilon_2 = 80 \times 10^{-6} \quad \text{and} \quad \theta_{p_2} = 109.0°$$

Note that $\epsilon_1 + \epsilon_2 = \epsilon_x + \epsilon_y$.

The principal strains are portrayed in Fig. 7-37c. There are, of course, no shear strains on the principal planes.

continued

(c) *Maximum shear strain*. The maximum shear strain is calculated from Eq. (7-75):

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 146 \times 10^{-6} \qquad \gamma_{max} = 290 \times 10^{-6}$$

The element having the maximum shear strains is oriented at 45° to the principal directions; therefore, $\theta_s = 19.0° + 45° = 64.0°$ and $2\theta_s = 128.0°$. By substituting this value of $2\theta_s$ into the second transformation equation (Eq. 7-71b), we can determine the sign of the shear strain associated with this direction. The calculations are as follows:

$$\frac{\gamma_{x_1y_1}}{2} = -\frac{\epsilon_x - \epsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

$$= -(115 \times 10^{-6})(\sin 128.0°) + (90 \times 10^{-6})(\cos 128.0°)$$

$$= -146 \times 10^{-6}$$

This result shows that an element oriented at an angle $\theta_{s_2} = 64.0°$ has the maximum negative shear strain.

We can arrive at the same result by observing that the angle θ_{s_1} to the direction of maximum positive shear strain is always 45° less than θ_{p_1}. Hence,

$$\theta_{s_1} = \theta_{p_1} - 45° = 19.0° - 45° = -26.0°$$

$$\theta_{s_2} = \theta_{s_1} + 90° = 64.0°$$

The shear strains corresponding to θ_{s_1} and θ_{s_2} are $\gamma_{max} = 290 \times 10^{-6}$ and $\gamma_{min} = -290 \times 10^{-6}$, respectively.

The normal strains on the element having the maximum and minimum shear strains are

$$\epsilon_{aver} = \frac{\epsilon_x + \epsilon_y}{2} = 225 \times 10^{-6}$$

A sketch of the element having the maximum in-plane shear strains is shown in Fig. 7-37d.

In this example, we solved for the strains by using the transformation equations. However, all of the results can be obtained just as easily from Mohr's circle.

Example 7-8

A 45° strain rosette (also called a *rectangular rosette*) consists of three electrical-resistance strain gages arranged to measure strains in two perpendicular directions and also at a 45° angle between them, as shown in Fig. 7-38a. The rosette is bonded to the surface of the structure before it is loaded. Gages A, B, and C measure the normal strains ϵ_a, ϵ_b, and ϵ_c in the directions of lines Oa, Ob, and Oc, respectively.

Explain how to obtain the strains ϵ_{x_1}, ϵ_{y_1}, and $\gamma_{x_1 y_1}$ associated with an element oriented at an angle θ to the xy axes (Fig. 7-38b).

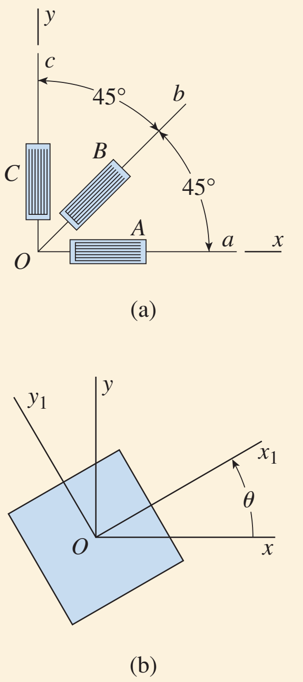


FIG. 7-38 Example 7-8. (a) 45° strain rosette, and (b) element oriented at an angle θ to the xy axes

Solution

At the surface of the stressed object, the material is in plane stress. Since the strain-transformation equations (Eqs. 7-71a and 7-71b) apply to plane stress as well as to plane strain, we can use those equations to determine the strains in any desired direction.

Strains associated with the xy axes. We begin by determining the strains associated with the xy axes. Because gages A and C are aligned with the x and y axes, respectively, they give the strains ϵ_x and ϵ_y directly:

$$\epsilon_x = \epsilon_a \qquad \epsilon_y = \epsilon_c \tag{7-77a,b}$$

To obtain the shear strain γ_{xy}, we use the transformation equation for normal strains (Eq. 7-71a):

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

For an angle $\theta = 45°$, we know that $\epsilon_{x_1} = \epsilon_b$ (Fig. 7-38a); therefore, the preceding equation gives

$$\epsilon_b = \frac{\epsilon_a + \epsilon_c}{2} + \frac{\epsilon_a - \epsilon_c}{2}(\cos 90°) + \frac{\gamma_{xy}}{2}(\sin 90°)$$

Solving for γ_{xy}, we get

$$\gamma_{xy} = 2\epsilon_b - \epsilon_a - \epsilon_c \tag{7-78}$$

Thus, the strains ϵ_x, ϵ_y, and γ_{xy} are easily determined from the given strain-gage readings.

Strains associated with the x_1y_1 axes. Knowing the strains ϵ_x, ϵ_y, and γ_{xy}, we can calculate the strains for an element oriented at any angle θ (Fig. 7-38b) from the strain-transformation equations (Eqs. 7-71a and 7-71b) or from Mohr's circle. We can also calculate the principal strains and the maximum shear strains from Eqs. (7-74) and (7-75), respectively.