Real numbers

Types of Real Numbers $\mathbb{R}$

1. **Natural numbers** $\mathbb{N} = \{1,2,3,4,\cdots +\}$
2. **Integer numbers** $\mathbb{Z} = \{0,\mp1,\mp2,\mp3,\mp4,\cdots\}$
3. **Rational numbers** $\mathbb{Q}$

$$r = \frac{m}{n} \ : m \in Z , n \in N$$

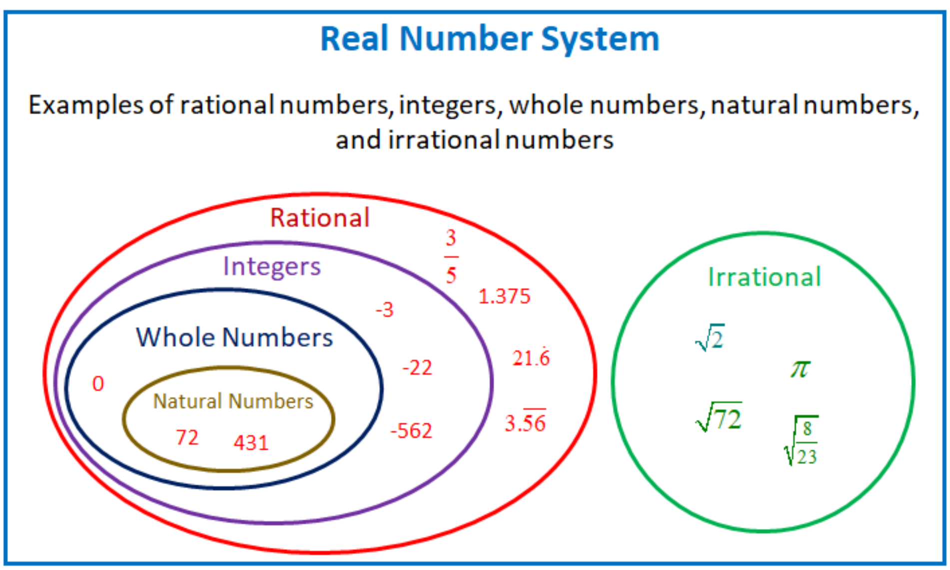
Examples

$$\frac{1}{2},\frac{5}{3},-\frac{16}{5},4 = \frac{16}{4},0.2 = \frac{2}{10}$$

4. **Irrational Numbers**: These are numbers that cannot be expressed as a ratio of integers

Examples

$$\sqrt{2},\sqrt[3]{5},\pi,2^{\sqrt{3}},e$$

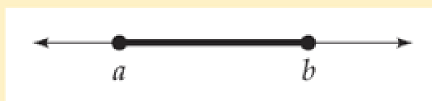
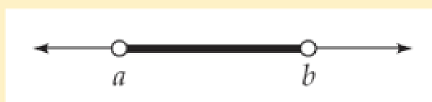
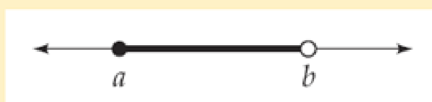
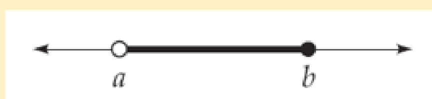


Intervals

Certain sets of real numbers, called intervals, occur frequently in calculus and correspond geometrically to line segments. If $a < b$, then the open interval from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). The closed interval from a to b includes the endpoints and is denoted $[a, b]$. Using set-builder notation, we can write

$$(a, b) = \{ x : a < x < b \}, [a, b] = \{ x : a \leq x \leq b \}$$

Finite Intervals

Interval Notation	Set Notation	Graph	Type
$[a, b]$	$\{ x \mid a \leq x \leq b \}$		Closed
(a, b)	$\{ x \mid a < x < b \}$		Open
$[a, b)$	$\{ x \mid a \leq x < b \}$		Half-open or half-closed
$(a, b]$	$\{ x \mid a < x \leq b \}$		Half-open or half-closed

Absolute Value: The absolute value of a number a, denoted by $|a|$.

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$$

Evenness: $|x| = |-x|$ example: $|4| = |-4|$

Subadditivity: $|x+y| \leq |x|+|y|$ example: $|4+1| \leq |4|+|1|$

Triangle Equality: $|x-y| \leq |x-a|+|a-y|$ example: $|4-1| \leq |4-2|+|2-1|$

Multiplicativeness: $|x * y| = |x| * |y|$ example: $|3*5| = |3|*|5|$

Preservation of division: $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ where $y \neq 0$ example: $\left|\frac{9}{3}\right| = \frac{|9|}{|3|}$

Inequalities:

An inequality is an expression involving one of the symbols $\geq, \leq, >$ or $<$ When we are asked to solve an inequality, the inequality will contain an unknown variable, say x. Solving means obtaining all values of x for which the inequality is true.

Properties:

1. Adding or subtracting the same quantity from both sides of an inequality leaves the inequality sign unchanged. ($If\ x < a$ ***then*** $x \mp b < a \mp b$.)

2. Multiplying or dividing both sides by a positive number leaves the inequality sign unchanged.

$$\text{If } x < a \text{ and } b > 0 \textbf{ then } xb < ab \textbf{ and } \frac{x}{b} < \frac{a}{b}$$

3. Multiplying or dividing both sides by a negative number reverses the inequality.

$$\text{If } x < a \text{ and } b < 0 \textbf{ then } xb > ab \textbf{ and } \frac{X}{b} > \frac{a}{b}$$

Example 1: Solve $3x - 5 \le 13$.

$$3x - 5 \le 13 \Rightarrow 3x \le 18 \Rightarrow x \le 6$$

Example 2: Solve $-2x - 4 \le -10$.

$$-2x - 4 \le -10 \Rightarrow -2x \le -6 \Rightarrow x \ge 3$$

Example 3: Solve $|2x - 5| \le 7$.

$$|2x - 5| < 7 \Rightarrow -7 < 2x - 5 < 7$$

$$\Rightarrow -7 + 5 < 2x < 7 + 5$$

$$\Rightarrow -2 < 2x < 12$$

$$\Rightarrow -1 < x < 6$$

Example 4: Solve $|2 - 5x| \ge 3$.

$$|2 - 5x| \ge 3 \Rightarrow 2 - 5x \ge 3 \text{ or } 2 - 5x \le -3$$

$$\Rightarrow -5x \ge 1 \text{ or } -5x \le -5$$

$$\Rightarrow x \le \frac{-1}{5} \text{ or } x \ge 1$$

The solution is $\{x: x \le -\frac{1}{5}\} \cup \{x: x \ge 1\}$

Solve quadratic inequality:

A quadratic inequality can be written in one of the standard forms

$ax^2 + bx + c < 0$ $\quad ax^2 + bx + c > 0$ $\quad ax^2 + bx + c \le 0$ $\quad ax^2 + bx + c \ge 0$

where a , b and c are real numbers and $a \neq 0$.

To solve a quadratic inequality in one variable, we will use the following steps to find the values of the variable that make the inequality true.

1- Write the inequality in standard form and solve its related quadratic equation.
2- Locate the solutions (called critical numbers) of the related quadratic equation on a number- line.
3- Test each interval on the number line created in step 2 by choosing a test value from the interval and determining whether it satisfies the inequality. The solution set includes the intervals whose test value make the inequality true.
4- Determine whether the endpoints of the intervals are included in the solution set.

Example 5: Solve $x^2 + x < 6$.

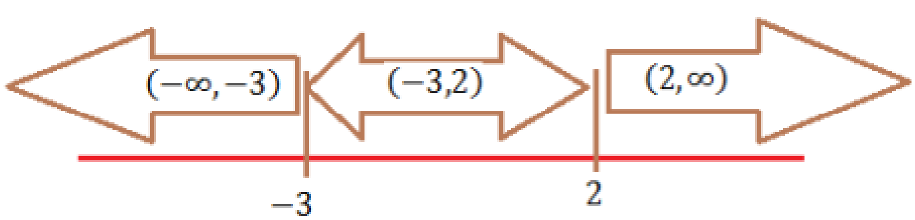
Solution: $x^2 + x < 6 \Rightarrow x^2 + x - 6 < 0$

We will solve the related quadratic equation $x^2 + x - 6 = 0$

$(x-2)(x+3) = 0 \quad \Rightarrow x = 2 , -3$

These two critical numbers will separate the number-line into three intervals

$(-\infty, -3)$ $(-3,2)$ $(2, \infty)$



If we choose $-4 \in (-\infty, -3) \Rightarrow (-4) + (-4) - 6 < 0 \Rightarrow 6 < 0$ False ×

If we choose $0 \in (-3,2) \Rightarrow (0) + 0 - 6 < 0 \Rightarrow -6 < 0$ True √

If we choose $3 \in (2, \infty) \Rightarrow (3) + 3 - 6 < 0 \Rightarrow 6 < 0$ False ×

Then the solution is the interval **$(-3,2)$**

Example 6: Solve $15 + 2x - x^2 \leq 0$.

Solution: We will solve the related quadratic equation $15 + 2x - x^2 = 0$.

$x^2 - 2x - 15 = 0 \Rightarrow (x - 5)(x + 3) = 0 \Rightarrow x = 5 , -3$

These two critical numbers will separate the number-line into three intervals

$(-\infty, -3]$,$[-3,5]$ and $[5, \infty)$

If we choose $-4 \in (-\infty, -3] \Rightarrow 15 + 2 \times (-4) - (-4)^2 \leq 0 \Rightarrow -9 \leq 0$ True √

If we choose $0 \in [-3,5] \Rightarrow 15 + 2 \times 0 - (0)^2 \leq 0 \Rightarrow 15 \leq 0$ False ×

If we choose $6 \in [5, \infty) \Rightarrow 15 + 2 \times 6 - (6)^2 \leq 0 \Rightarrow -9 \leq 0$ True √

Then the solution is $(-\infty, -3] \cup [5, \infty)$

Real functions

A function is a relation that uniquely associates members of one set with members of another set. A function whose range is in the real numbers $\mathbb{R}$ is said to be a real function, also called a real-valued function $f : \mathbb{R} \to \mathbb{R}$.

The domain of a function is the set of all the numbers you can substitute in to the function (x - values). The range of a function is the set of all the numbers you can get out of the function (y - values).

When determining the domain of a function from a formula, we really only have to look out for two situations:

1- Rational expression (fractions) – Division by zero is not allowed so we must omit any values of x which make the denominator zero.
2- Even roots – For even roots such as square roots, the radicand cannot be negative. That is the radicand greater than or equal to zero.

Examples: Find the domain for the functions

1. $f(x) = \frac{5x}{x^2 - 3x - 4}$

Since this is a rational expression, we must not let the denominator equal zero.

What values of x make denominator equal zero?

$x^2 - 3x - 4 = 0 \Rightarrow (x-4)(x+1) = 0 \Rightarrow x = 4$ or $x = -3$

Thus the domain is $D = \mathbb{R}/\{4, -3\}$

2. $f(x) = \frac{2x}{\sqrt{3x - 4}}$

We must be very careful with this function since it involves both rational expression and square root. The square root requires the radicand be greater than or equal to zero, that is, $3x - 4 \geq 0$. However , since the square root is in the denominator and we cannot divide by zero.

Thus $3x - 4 > 0 \Rightarrow 3x > 4$, then the domain is

$D = (\frac{3}{4}, \infty)$