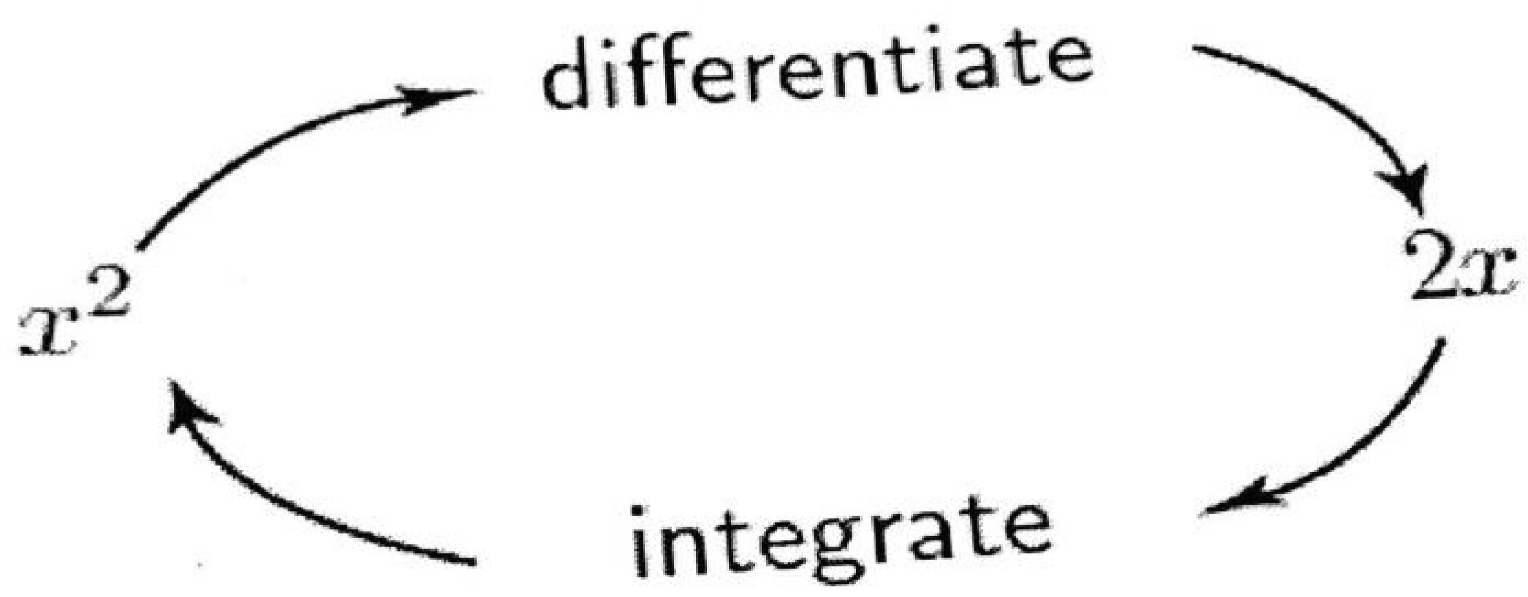
Basic Concepts of Integration

Introduction

When a function $f(x)$ is known we can differentiate it to obtain its derivative $\frac{df}{dx}$. The reverse process is to obtain the function $f(x)$ from knowledge of its derivative. This process is called **integration**. Applications of integration are numerous and some of these will be explored in subsequent Sections. First, what is important is to practise basic techniques and learn a variety of methods for integrating functions.



$$\int \underbrace{2x}_{} \, dx = x^2 + c$$

integral sign

this term is called the integrand

there must always be a term of the form dx

constant of integration

Integrals of Functions

function $f(x)$	indefinite integral $\int f(x)\,dx$
constant, k	$kx + c$
x	$\frac{1}{2}x^2 + c$
x^2	$\frac{1}{3}x^3 + c$
x^n	$\dfrac{x^{n+1}}{n+1} + c, \quad n \neq -1$
x^{-1} (or $\frac{1}{x}$)	$\ln\lvert x\rvert + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\cos kx$	$\dfrac{1}{k}\sin kx + c$
$\sin kx$	$-\dfrac{1}{k}\cos kx + c$
$\tan kx$	$\dfrac{1}{k}\ln\lvert\sec kx\rvert + c$
e^x	$e^x + c$
e^{-x}	$-e^{-x} + c$
e^{kx}	$\dfrac{1}{k}e^{kx} + c$

TRIGONOMETRIC DERIVATIVES

$\frac{d}{dx}(\sin(x)) = \cos(x)\cdot x'$	$\frac{d}{dx}(\cos(x)) = -\sin(x)\cdot x'$	$\frac{d}{dx}(\tan(x)) = \sec^2(x)\cdot x'$
$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)\cdot x'$	$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)\cdot x'$	$\frac{d}{dx}(\cot(x)) = -\csc^2(x)\cdot x'$
$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}\cdot x'$	$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}\cdot x'$	$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}\cdot x'$
$\frac{d}{dx}(\csc^{-1}(x)) = -\frac{1}{x\sqrt{x^2-1}}\cdot x'$	$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2-1}}\cdot x'$	$\frac{d}{dx}(\cot^{-1}(x)) = -\frac{1}{1+x^2}\cdot x'$

TRIGONOMETRIC INTEGRALS

$\int \sin(x)dx = -\cos(x) + C$	$\int \csc(x)dx = \ln\lvert\csc(x) - \cot(x)\rvert + C$
$\int \cos(x)dx = \sin(x) + C$	$\int \sec(x)dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$
$\int \tan(x)dx = \ln\lvert\sec(x)\rvert + C$	$\int \cot(x)dx = \ln\lvert\sin(x)\rvert + C$

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

INVERSE TRIG INTEGRALS

$\int \sin^{-1}(x)dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$
$\int \cos^{-1}(x)dx = x\cos^{-1}(x) - \sqrt{1-x^2} + C$
$\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

Example 1

Use Table 1 to find the indefinite integral of x^7: that is, find $\int x^7\,dx$

Solution

From Table 1 note that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$. In words, this states that to integrate a po
of x, increase the power by 1, and then divide the result by the new power. With $n = 7$ we fin

$$\int x^7\,dx = \frac{1}{8}x^8 + c$$

Example

$$\int \cos^2\theta \, d\theta = \int \frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C.$$

Example

Find

1 $\int_0^{\frac{\pi}{2}} (1+\cos 2x)\, dx$

2 $\int_0^{\frac{\pi}{6}} (\sin 3x + \sec^2 2x)\, dx.$

Solution

1 $\int_0^{\frac{\pi}{2}} (1+\cos 2x)\, dx = \left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

2 $\int_0^{\frac{\pi}{6}} (\sin 3x + \sec^2 2x)\, dx = \left[-\frac{1}{3}\cos 3x + \frac{1}{2}\tan 2x\right|_0^{\frac{\pi}{6}} = \frac{1}{3} + \frac{\sqrt{3}}{2}.$

Integration of Hyperbolic Functions

$(\sinh x)' = \cosh x$	$\int \cosh x\, dx = \sinh x + C$
$(\cosh x)' = \sinh x$	$\int \sinh x\, dx = \cosh x + C$
$(\tanh x)' = \operatorname{sech}^2 x$	$\int \operatorname{sech}^2 x\, dx = \tanh x + C$
$(\coth x)' = -\operatorname{csch}^2 x$	$\int \operatorname{csch}^2 x\, dx = -\coth x + C$
$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$	$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$
$(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$	$\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$

Integration By Substitution

التكامل بالتعويض

الفكرة :- نفرض (Z) اقتراناً مشتقته موجودة في الجواب، او لها علاقة بالجواب، أي شبه تجميع مشتقة

* أي مشتقتها لدالة موجودة بجوار الدالة

* $\int 5x^4 \cdot (x^5+2)^6 \, dx$ ← لا يحل بالطريقة العادية

طريقة الحل

① نشتق $(x^5+2)' = 5x^4 + 0$

∴ مشتقة موجودة بالتالي نفرض

$$\boxed{Z = x^5 + 2}$$

② نشتق كل طرف

$1 \, dZ = 5x^4 \, dx$ ← نجد منها x نقسم كل الطرفين على المعامل (dx)

$$\frac{1}{5x^4} dZ = \frac{\cancel{5x^4}}{\cancel{5x^4}} dx$$

$$\boxed{dx = \frac{dZ}{5x^4}}$$

∴ مفتاح الحل // ايجاد (Z) و dx

③ نرجع للسؤال ونعوض // (dx) و (Z)

$$\int 5x^4 \cdot (x^5+2)^6 \, dx = \int \cancel{5x^4} \cdot (Z)^6 \cdot \frac{dZ}{\cancel{5x^4}}$$

$$= \int Z^6 \, dZ = \frac{Z^7}{7} + C$$

④ نرجع الجواب بدلالة x

$$\frac{(x^5+2)^7}{7} + C$$

★ الحالات

١) الاقتران مرفوع ع الى قوة او ماداخل القوس

$$\int 2x(x^2+1)^7\,dx \Rightarrow z = x^2+1$$

٢) ماداخل الجذر او الجذر نفسه

$$\int x\cdot\sqrt{x^2+1}\,dx \rightarrow z = x^2+1$$

اما الذي داخل الجذر

او

الجذر نفسه

or $z = \sqrt{x^2+1}$

٣) زاوية الاقتران الدائري غير الخطية

الخطية $\underline{ax+b}$

$$\int 3x^2\cdot\cos(x^3+1)\,dx \Rightarrow z = x^3+1$$

٤) قوة الاقتران الأسي غير الخطية

$$\int \cos x\cdot e^{(\sin x)}\,dx \Rightarrow z = \sin x$$

٥) اقتران ال Inverse

$$\int \frac{\sqrt{\tan^{-1}x}}{x^2+1} \Rightarrow z = \tan^{-1}x$$

or $z = \sqrt{\tan^{-1}x}$

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Examples

① $\int_0^4 x \cdot \sqrt{x^2+9}\, dx$ $\quad z =$ ما داخل الجذر او الجذر نفسه

بالترجيح ① $\boxed{z = \sqrt{x^2+9}}$

$$z^2 = x^2 + 9$$

② نشتق $\cancel{2} z\, dz = \cancel{2} x \cdot dx$ / نقسم على x

③ $\boxed{dx = \frac{z\, dz}{x}}$

Note نلاحظ حدود التكامل $\int_0^4$ سوف نغيرها حسب قيمة (z) من خلال تعويض قيمة x

at $x = 0 \rightarrow z = \sqrt{0+9} = 3$

at $x = 4 \rightarrow z = \sqrt{16+9} = \sqrt{25} = 5$

④ $\int_3^5 \cancel{x} \cdot z \cdot \frac{z\, dz}{\cancel{x}} \Rightarrow \int_3^5 z^2\, dz = \left. \frac{z^3}{3} \right|_3^5$

$$= \frac{125}{3} - \frac{27}{3} = \boxed{\frac{98}{3}}$$

② $\int x^3 \sin(\underline{2x^4})\, dx$

غير خطية

$$\boxed{Z = 2x^4}$$

① نشتق $dz = 8x^3 \cdot dx$ / نقسم على $8x^3$

$$\therefore \boxed{dx = \frac{dZ}{8x^3}}$$

$$\Rightarrow \int \cancel{x^3} \sin(Z) \cdot \frac{dZ}{8\cancel{x^3}}$$

$$= \frac{1}{8} \int \sin(Z) \cdot dZ = \frac{1}{8}\left(-\cos(Z)\right) + C$$

$$= \frac{-1}{8} \cdot \cos(2x^4) + C$$

③ $\int \frac{(\ln x)^3}{x}\, dx$

Note

مشتقة $\boxed{\ln x = \frac{1}{x}}$

∴ المشتقة موجودة

① $\therefore Z = \ln x$

② $dZ = \frac{1}{x} \cdot dx \longrightarrow$ نضرب كلا الطرفين في x

$$\boxed{dx = x \cdot dZ}$$

$$\therefore \int \frac{(Z)^3}{\cancel{x}} \cdot \cancel{x}\, dZ = \int Z^3 \cdot dZ$$

$$= \frac{Z^4}{4} + C$$

$$= \frac{(\ln x)^4}{4} + C$$