In terms of vector-matrix equations, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{M+m}{Ml}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{m}{M}g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{1}{Ml} \\ 0 \\ \dfrac{1}{M} \end{bmatrix} u \tag{3–22}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{3–23}$$

Equations (3–22) and (3–23) give a state-space representation of the inverted-pendulum system. (Note that state-space representation of the system is not unique. There are infinitely many such representations for this system.)

3–3 MATHEMATICAL MODELING OF ELECTRICAL SYSTEMS

Basic laws governing electrical circuits are Kirchhoff's current law and voltage law. Kirchhoff's current law (node law) states that the algebraic sum of all currents entering and leaving a node is zero. (This law can also be stated as follows: The sum of currents entering a node is equal to the sum of currents leaving the same node.) Kirchhoff's voltage law (loop law) states that at any given instant the algebraic sum of the voltages around any loop in an electrical circuit is zero. (This law can also be stated as follows: The sum of the voltage drops is equal to the sum of the voltage rises around a loop.) A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's laws to it.

This section first deals with simple electrical circuits and then treats mathematical modeling of operational amplifier systems.

***LRC* Circuit.** Consider the electrical circuit shown in Figure 3–7. The circuit consists of an inductance L (henry), a resistance R (ohm), and a capacitance C (farad). Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \tag{3–24}$$

$$\frac{1}{C}\int i\,dt = e_o \tag{3–25}$$

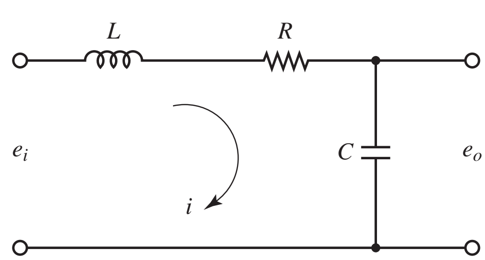


Figure 3–7
Electrical circuit.

Equations (3–24) and (3–25) give a mathematical model of the circuit.

A transfer-function model of the circuit can also be obtained as follows: Taking the Laplace transforms of Equations (3–24) and (3–25), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

If e_i is assumed to be the input and e_o the output, then the transfer function of this system is found to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \tag{3–26}$$

A state-space model of the system shown in Figure 3–7 may be obtained as follows: First, note that the differential equation for the system can be obtained from Equation (3–26) as

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Then by defining state variables by

$$x_1 = e_o$$
$$x_2 = \dot{e}_o$$

and the input and output variables by

$$u = e_i$$
$$y = e_o = x_1$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

and

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a mathematical model of the system in state space.

Transfer Functions of Cascaded Elements. Many feedback systems have components that load each other. Consider the system shown in Figure 3–8. Assume that e_i is the input and e_o is the output. The capacitances C_1 and C_2 are not charged initially.

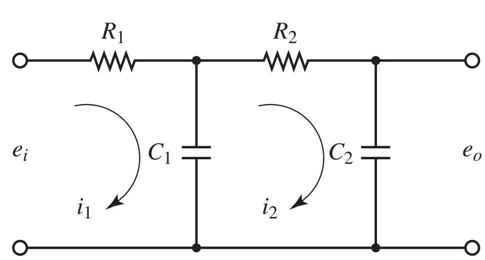


Figure 3–8
Electrical system.

It will be shown that the second stage of the circuit (R_2C_2 portion) produces a loading effect on the first stage (R_1C_1 portion). The equations for this system are

$$\frac{1}{C_1}\int(i_1 - i_2)\,dt + R_1 i_1 = e_i \tag{3–27}$$

and

$$\frac{1}{C_1}\int(i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0 \tag{3–28}$$

$$\frac{1}{C_2}\int i_2\,dt = e_o \tag{3–29}$$

Taking the Laplace transforms of Equations (3–27) through (3–29), respectively, using zero initial conditions, we obtain

$$\frac{1}{C_1 s}\left[I_1(s) - I_2(s)\right] + R_1 I_1(s) = E_i(s) \tag{3–30}$$

$$\frac{1}{C_1 s}\left[I_2(s) - I_1(s)\right] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0 \tag{3–31}$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s) \tag{3–32}$$

Eliminating $I_1(s)$ from Equations (3–30) and (3–31) and writing $E_i(s)$ in terms of $I_2(s)$, we find the transfer function between $E_o(s)$ and $E_i(s)$ to be

$$\begin{aligned}\frac{E_o(s)}{E_i(s)} &= \frac{1}{(R_1C_1s + 1)(R_2C_2s + 1) + R_1C_2s} \\ &= \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}\end{aligned} \tag{3–33}$$

The term R_1C_2s in the denominator of the transfer function represents the interaction of two simple RC circuits. Since $(R_1C_1 + R_2C_2 + R_1C_2)^2 > 4R_1C_1R_2C_2$, the two roots of the denominator of Equation (3–33) are real.

The present analysis shows that, if two RC circuits are connected in cascade so that the output from the first circuit is the input to the second, the overall transfer function is not the product of $1/(R_1C_1s + 1)$ and $1/(R_2C_2s + 1)$. The reason for this is that, when we derive the transfer function for an isolated circuit, we implicitly assume that the output is unloaded. In other words, the load impedance is assumed to be infinite, which means that no power is being withdrawn at the output. When the second circuit is connected to the output of the first, however, a certain amount of power is withdrawn, and thus the assumption of no loading is violated. Therefore, if the transfer function of this system is obtained under the assumption of no loading, then it is not valid. The degree of the loading effect determines the amount of modification of the transfer function.

Complex Impedances. In deriving transfer functions for electrical circuits, we frequently find it convenient to write the Laplace-transformed equations directly, without writing the differential equations. Consider the system shown in Figure 3–9(a). In this system, Z_1 and Z_2 represent complex impedances. The complex impedance $Z(s)$ of a two-terminal circuit is the ratio of $E(s)$, the Laplace transform of the voltage across the terminals, to $I(s)$, the Laplace transform of the current through the element, under the assumption that the initial conditions are zero, so that $Z(s) = E(s)/I(s)$. If the two-terminal element is a resistance R, capacitance C, or inductance L, then the complex impedance is given by R, $1/Cs$, or Ls, respectively. If complex impedances are connected in series, the total impedance is the sum of the individual complex impedances.

Remember that the impedance approach is valid only if the initial conditions involved are all zeros. Since the transfer function requires zero initial conditions, the impedance approach can be applied to obtain the transfer function of the electrical circuit. This approach greatly simplifies the derivation of transfer functions of electrical circuits.

Consider the circuit shown in Figure 3–9(b). Assume that the voltages e_i and e_o are the input and output of the circuit, respectively. Then the transfer function of this circuit is

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

For the system shown in Figure 3–7,

$$Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs}$$

Hence the transfer function $E_o(s)/E_i(s)$ can be found as follows:

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

which is, of course, identical to Equation (3–26).

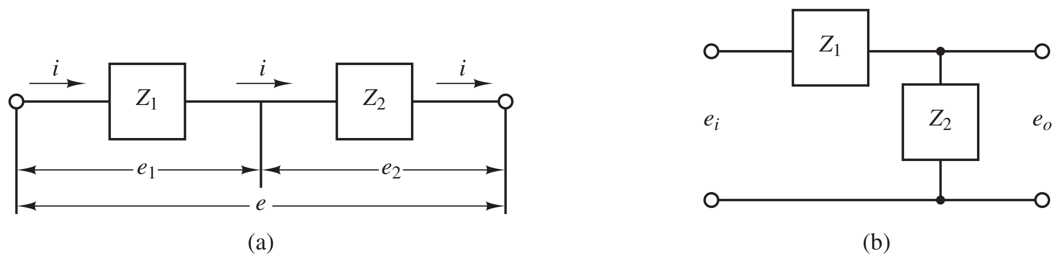


Figure 3–9
Electrical circuits.

EXAMPLE 3–7 Consider again the system shown in Figure 3–8. Obtain the transfer function $E_o(s)/E_i(s)$ by use of the complex impedance approach. (Capacitors C_1 and C_2 are not charged initially.)

The circuit shown in Figure 3–8 can be redrawn as that shown in Figure 3–10(a), which can be further modified to Figure 3–10(b).

In the system shown in Figure 3–10(b) the current I is divided into two currents I_1 and I_2. Noting that

$$Z_2 I_1 = (Z_3 + Z_4) I_2, \qquad I_1 + I_2 = I$$

we obtain

$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

Noting that

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} \right] I$$

$$E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$

we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}$$

Substituting $Z_1 = R_1$, $Z_2 = 1/(C_1 s)$, $Z_3 = R_2$, and $Z_4 = 1/(C_2 s)$ into this last equation, we get

$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{1}{C_1 s}\dfrac{1}{C_2 s}}{R_1\left(\dfrac{1}{C_1 s} + R_2 + \dfrac{1}{C_2 s}\right) + \dfrac{1}{C_1 s}\left(R_2 + \dfrac{1}{C_2 s}\right)}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

which is the same as that given by Equation (3–33).

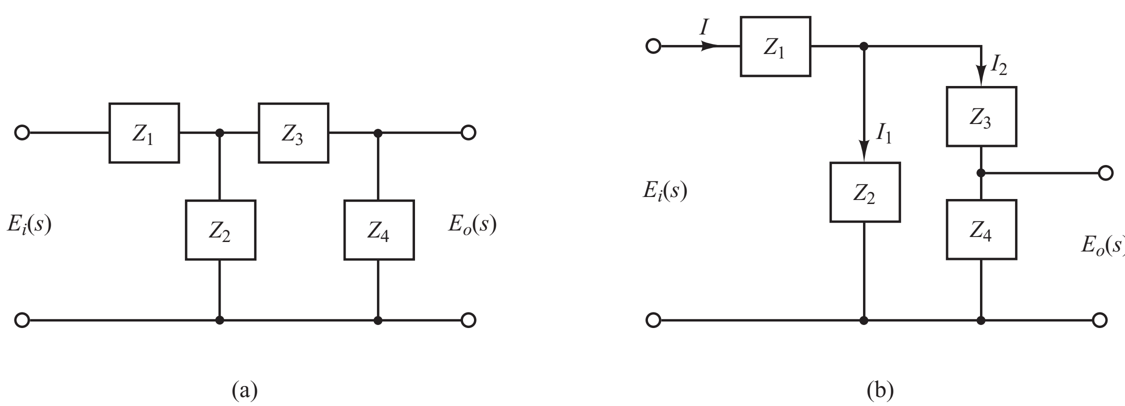


Figure 3–10 (a) The circuit of Figure 3–8 shown in terms of impedances; (b) equivalent circuit diagram.

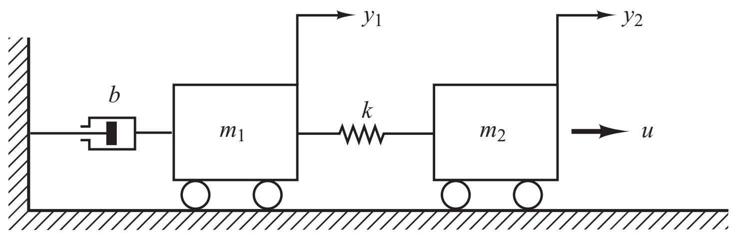


Figure 3–22
Mechanical system.

A–3–3. Obtain a state-space representation of the system shown in Figure 3–22.

Solution. The system equations are

$$m_1\ddot{y}_1 + b\dot{y}_1 + k(y_1 - y_2) = 0$$
$$m_2\ddot{y}_2 + k(y_2 - y_1) = u$$

The output variables for this system are y_1 and y_2. Define state variables as

$$x_1 = y_1$$
$$x_2 = \dot{y}_1$$
$$x_3 = y_2$$
$$x_4 = \dot{y}_2$$

Then we obtain the following equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m_1}\left[-b\dot{y}_1 - k(y_1 - y_2)\right] = -\frac{k}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k}{m_1}x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{1}{m_2}\left[-k(y_2 - y_1) + u\right] = \frac{k}{m_2}x_1 - \frac{k}{m_2}x_3 + \frac{1}{m_2}u$$

Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k}{m_1} & -\dfrac{b}{m_1} & \dfrac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{m_2} & 0 & -\dfrac{k}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix} u$$

and the output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

A–3–4. Obtain the transfer function $X_o(s)/X_i(s)$ of the mechanical system shown in Figure 3–23(a). Also obtain the transfer function $E_o(s)/E_i(s)$ of the electrical system shown in Figure 3–23(b). Show that these transfer functions of the two systems are of identical form and thus they are analogous systems.

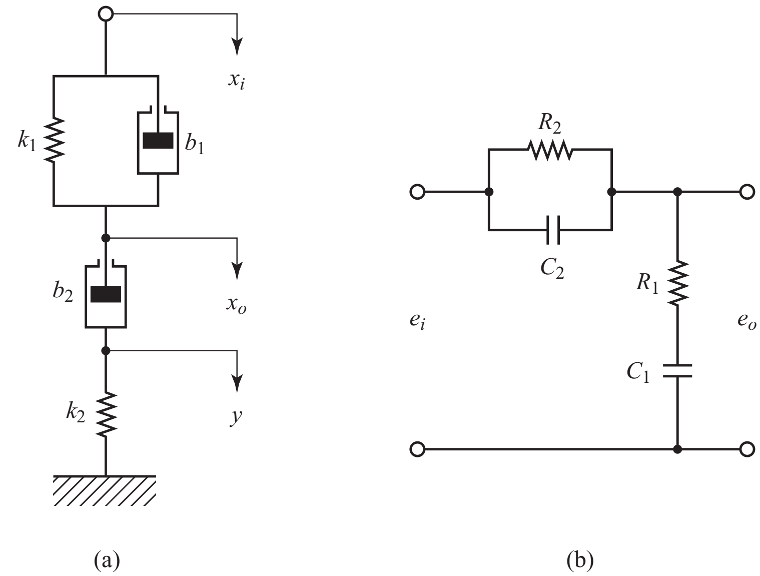


Figure 3–23
(a) Mechanical system;
(b) analogous electrical system.

Solution. In Figure 3–23(a) we assume that displacements x_i, x_o, and y are measured from their respective steady-state positions. Then the equations of motion for the mechanical system shown in Figure 3–23(a) are

$$b_1(\dot{x}_i - \dot{x}_o) + k_1(x_i - x_o) = b_2(\dot{x}_o - \dot{y})$$

$$b_2(\dot{x}_o - \dot{y}) = k_2 y$$

By taking the Laplace transforms of these two equations, assuming zero initial conditions, we have

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2[sX_o(s) - sY(s)]$$

$$b_2[sX_o(s) - sY(s)] = k_2 Y(s)$$

If we eliminate $Y(s)$ from the last two equations, then we obtain

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2 sX_o(s) - b_2 s \frac{b_2 sX_o(s)}{b_2 s + k_2}$$

or

$$(b_1 s + k_1)X_i(s) = \left(b_1 s + k_1 + b_2 s - b_2 s \frac{b_2 s}{b_2 s + k_2}\right)X_o(s)$$

Hence the transfer function $X_o(s)/X_i(s)$ can be obtained as

$$\frac{X_o(s)}{X_i(s)} = \frac{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right)}{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right) + \frac{b_2}{k_1}s}$$

For the electrical system shown in Figure 3–23(b), the transfer function $E_o(s)/E_i(s)$ is found to be

$$\frac{E_o(s)}{E_i(s)} = \frac{R_1 + \frac{1}{C_1 s}}{\frac{1}{(1/R_2) + C_2 s} + R_1 + \frac{1}{C_1 s}}$$

$$= \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_2 C_1 s}$$

A comparison of the transfer functions shows that the systems shown in Figures 3–23(a) and (b) are analogous.

A–3–5. Obtain the transfer functions $E_o(s)/E_i(s)$ of the bridged T networks shown in Figures 3–24(a) and (b).

Solution. The bridged T networks shown can both be represented by the network of Figure 3–25(a), where we used complex impedances. This network may be modified to that shown in Figure 3–25(b).

In Figure 3–25(b), note that

$$I_1 = I_2 + I_3, \qquad I_2 Z_1 = (Z_3 + Z_4)I_3$$

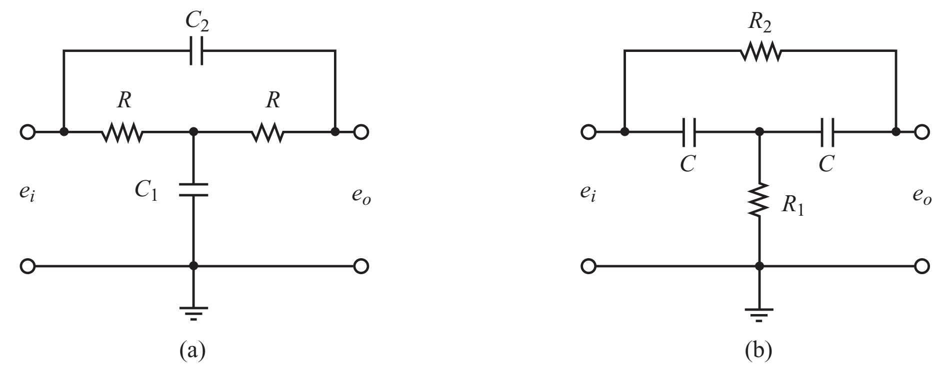


Figure 3–24
Bridged T networks.

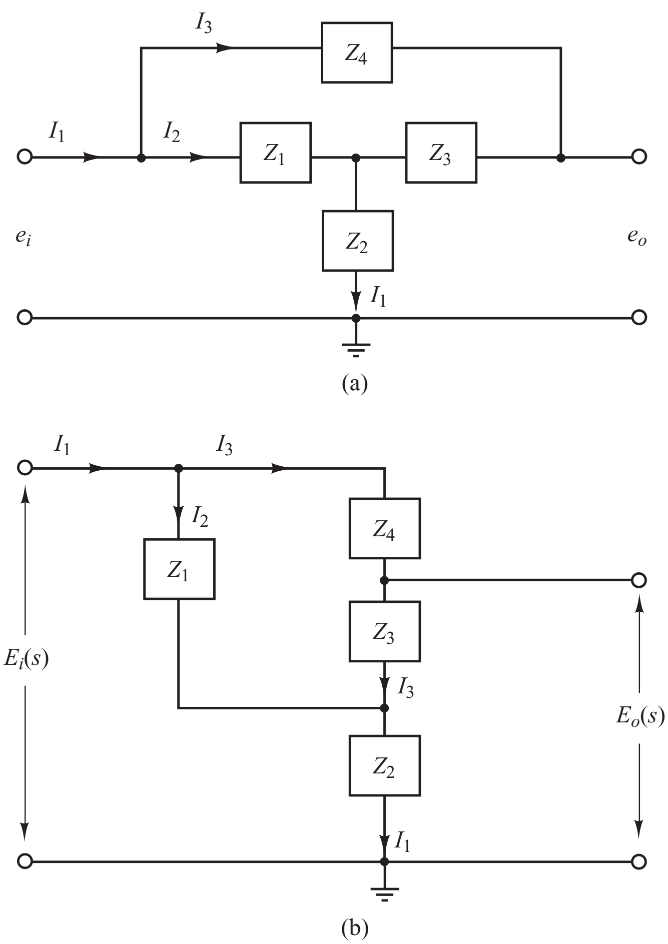


Figure 3–25
(a) Bridged T network in terms of complex impedances; (b) equivalent network.

Hence

$$I_2 = \frac{Z_3 + Z_4}{Z_1 + Z_3 + Z_4} I_1, \qquad I_3 = \frac{Z_1}{Z_1 + Z_3 + Z_4} I_1$$

Then the voltages $E_i(s)$ and $E_o(s)$ can be obtained as

$$\begin{aligned} E_i(s) &= Z_1 I_2 + Z_2 I_1 \\ &= \left[Z_2 + \frac{Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} \right] I_1 \\ &= \frac{Z_2(Z_1 + Z_3 + Z_4) + Z_1(Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} I_1 \\ E_o(s) &= Z_3 I_3 + Z_2 I_1 \\ &= \frac{Z_3 Z_1}{Z_1 + Z_3 + Z_4} I_1 + Z_2 I_1 \\ &= \frac{Z_3 Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_1 + Z_3 + Z_4} I_1 \end{aligned}$$

Hence, the transfer function $E_o(s)/E_i(s)$ of the network shown in Figure 3–25(a) is obtained as

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_3 Z_1 + Z_2(Z_1 + Z_3 + Z_4)}{Z_2(Z_1 + Z_3 + Z_4) + Z_1 Z_3 + Z_1 Z_4} \tag{3–38}$$

For the bridged T network shown in Figure 3–24(a), substitute

$$Z_1 = R, \qquad Z_2 = \frac{1}{C_1 s}, \qquad Z_3 = R, \qquad Z_4 = \frac{1}{C_2 s}$$

into Equation (3–38). Then we obtain the transfer function $E_o(s)/E_i(s)$ to be

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{R^2 + \frac{1}{C_1 s}\left(R + R + \frac{1}{C_2 s}\right)}{\frac{1}{C_1 s}\left(R + R + \frac{1}{C_2 s}\right) + R^2 + R\frac{1}{C_2 s}} \\ &= \frac{RC_1 RC_2 s^2 + 2RC_2 s + 1}{RC_1 RC_2 s^2 + (2RC_2 + RC_1)s + 1} \end{aligned}$$

Similarly, for the bridged T network shown in Figure 3–24(b), we substitute

$$Z_1 = \frac{1}{Cs}, \qquad Z_2 = R_1, \qquad Z_3 = \frac{1}{Cs}, \qquad Z_4 = R_2$$

into Equation (3–38). Then the transfer function $E_o(s)/E_i(s)$ can be obtained as follows:

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{\frac{1}{Cs}\frac{1}{Cs} + R_1\left(\frac{1}{Cs} + \frac{1}{Cs} + R_2\right)}{R_1\left(\frac{1}{Cs} + \frac{1}{Cs} + R_2\right) + \frac{1}{Cs}\frac{1}{Cs} + R_2\frac{1}{Cs}} \\ &= \frac{R_1 C R_2 C s^2 + 2R_1 C s + 1}{R_1 C R_2 C s^2 + (2R_1 C + R_2 C)s + 1} \end{aligned}$$

B–3–7. Obtain the transfer function $E_o(s)/E_i(s)$ of the electrical circuit shown in Figure 3–36.

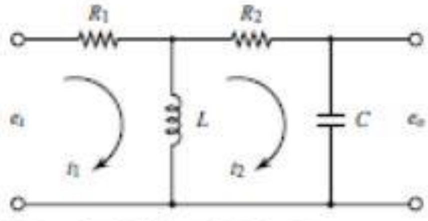


Figure 3–36 Electrical circuit.

B–3–8. Consider the electrical circuit shown in Figure 3–37. Obtain the transfer function $E_o(s)/E_i(s)$ by use of the block diagram approach.

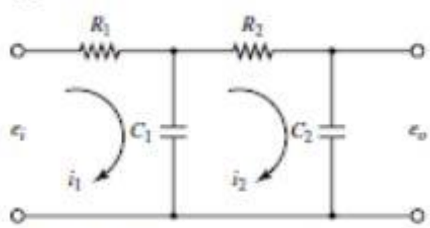


Figure 3–37 Electrical circuit.

B–3–9. Derive the transfer function of the electrical circuit shown in Figure 3–38. Draw a schematic diagram of an analogous mechanical system.

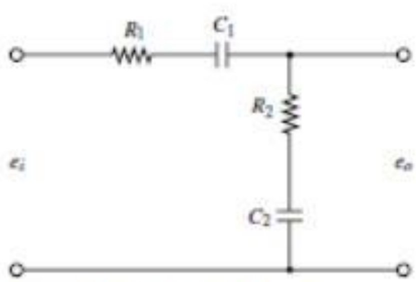


Figure 3–38 Electrical circuit.

Hence

$$\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + b_2 s + k_2 + k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}$$

From Equation (2), we obtain

$$\frac{X_2(s)}{X_1(s)} = \frac{k_3}{m_2 s^2 + b_2 s + k_2 + k_3}$$

Hence

$$\frac{X_2(s)}{U(s)} = \frac{X_2(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)} = \frac{k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}$$

B-3-7. The equations for the given circuit are as followw:

$$R_1 i_1 + L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = e_i$$

$$R_2 i_2 + \frac{1}{C}\int i_2\,dt + L\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) = 0$$

$$\frac{1}{C}\int i_2\,dt = e_o$$

Taking the Laplace transforms of these three equations, assuming zero initial conditions, gives

$$R_1 I_1(s) + L\left[s I_1(s) - s I_2(s)\right] = E_i(s) \qquad (1)$$

$$R_2 I_2(s) + \frac{1}{Cs} I_2(s) + L\left[s I_2(s) - s I_1(s)\right] = 0 \qquad (2)$$

$$\frac{1}{Cs} I_2(s) = E_o(s) \qquad (3)$$

From Equation (2) we obtain

$$\left(R_2 + \frac{1}{Cs} + Ls\right) I_2(s) = Ls\, I_1(s)$$

or

$$I_2(s) = \frac{LCs^2}{LCs^2 + R_2 C s + 1} I_1(s) \qquad (4)$$

Substituting Equation (4) into Equation (1), we get

$$\left(R_1 + Ls - Ls\,\frac{LCs^2}{LCs^2 + R_2 C s + 1}\right) I_1(s) = E_i(s)$$

or

$$\frac{LC(R_1 + R_2)s^2 + (R_1 R_2 C + L)s + R_1}{LCs^2 + R_2 C s + 1} I_1(s) = E_i(s) \qquad (5)$$

From Equations (3) and (4), we have

$$\frac{Ls}{LCs^2 + R_2 C s + 1} I_1(s) = E_o(s) \qquad (6)$$

From Equations (5) and (6), we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{Ls}{LC(R_1+R_2)s^2+(R_1R_2C+L)s+R_1}$$

B-3-8. Equations for the circuit are

$$\frac{1}{C_1}\int(i_1-i_2)\,dt + R_1 i_1 = e_i$$

$$\frac{1}{C_1}\int(i_2-i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0$$

$$\frac{1}{C_2}\int i_2\,dt = e_o$$

The Laplace transforms of these three equations, with zero initial conditions, are

$$\frac{1}{C_1 s}[I_1(s) - I_2(s)] + R_1 I_1(s) = E_i(s) \quad (1)$$

$$\frac{1}{C_1 s}[I_2(s) - I_1(s)] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0 \quad (2)$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s) \quad (3)$$

Equation (1) can be rewritten as

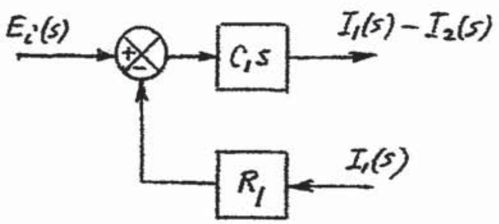
$$C_1 s[E_i(s) - R_1 I_1(s)] = I_1(s) - I_2(s) \quad (4)$$

Equation (4) gives the block diagram shown in Figure (a). Equation (2) can be modified to

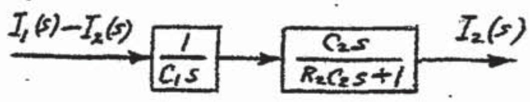
$$I_2(s) = \frac{C_2 s}{R_2 C_2 s + 1}\frac{1}{C_1 s}[I_1(s) - I_2(s)] \quad (5)$$

Equation (5) yields the block diagram shown in Figure (b). Also, Equation (3) gives the block diagram shown in Figure (c). Combining the block diagrams of Figures (a), (b), and (c), we obtain Figure (d). This block diagram can be succesively modified as shown in Figures (e) through (h). In this way, we can obtain the transfer function $E_o(s)/E_i(s)$ of the given circuit.

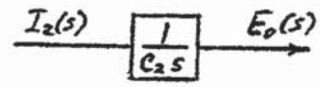
(a)


$E_i(s)$
$C_1 s$
$I_1(s) - I_2(s)$
R_1
$I_1(s)$

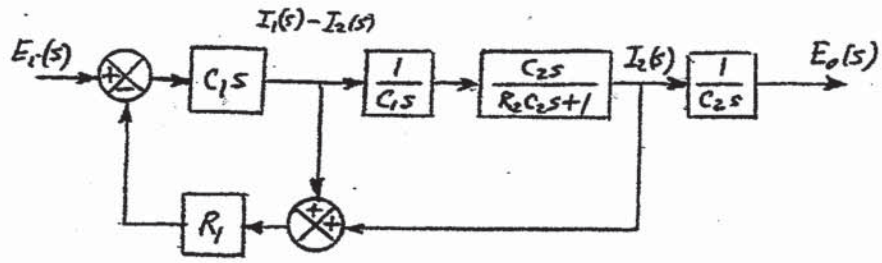
(b)


$I_1(s) - I_2(s)$
$\frac{1}{C_1 s}$
$\frac{C_2 s}{R_2 C_2 s + 1}$
$I_2(s)$

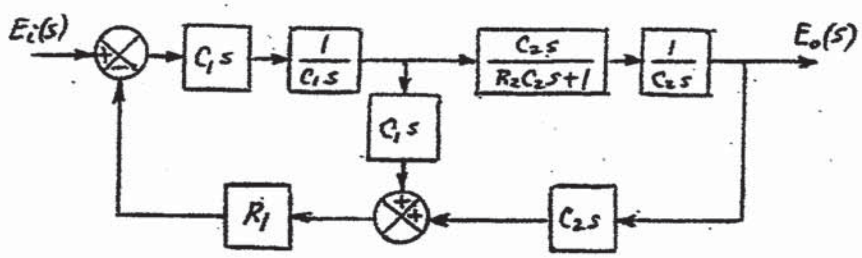
(c)


$I_2(s)$
$\frac{1}{C_2 s}$
$E_o(s)$

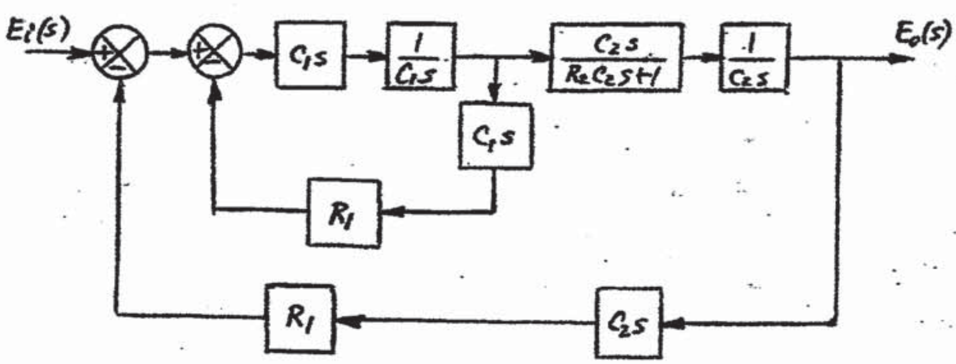
(d)


$E_i(s)$
$I_1(s) - I_2(s)$
$C_1 s$
$\frac{1}{C_1 s}$
$\frac{C_2 s}{R_2 C_2 s + 1}$
$I_2(s)$
$\frac{1}{C_2 s}$
$E_o(s)$
R_1

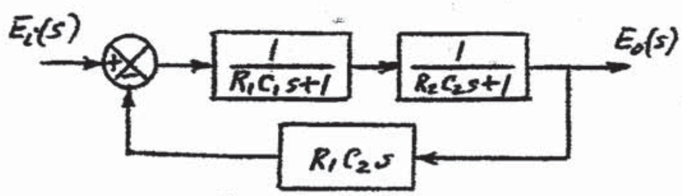
(e)


$E_i(s)$
$C_1 s$
$\frac{1}{C_1 s}$
$\frac{C_2 s}{R_2 C_2 s + 1}$
$\frac{1}{C_2 s}$
$E_o(s)$
$C_1 s$
R_1
$C_2 s$

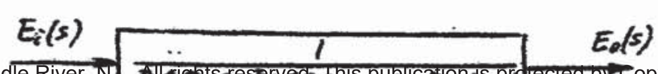
(f)


$E_i(s)$
$C_1 s$
$\frac{1}{C_1 s}$
$\frac{C_2 s}{R_2 C_2 s + 1}$
$\frac{1}{C_2 s}$
$E_o(s)$
$C_1 s$
R_1
R_1
$C_2 s$

(g)

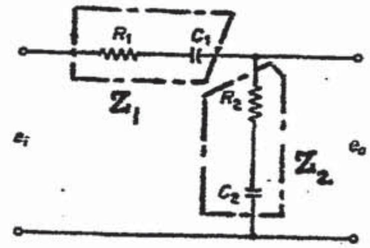

$E_i(s)$
$\frac{1}{R_1 C_1 s + 1}$
$\frac{1}{R_2 C_2 s + 1}$
$E_o(s)$
$R_1 C_2 s$

(h)


$E_i(s)$
$E_o(s)$

B-3-9. Impedance Z_1 is

$$Z_1 = R_1 + \frac{1}{C_1 s}$$



Impedance Z_2 is

$$Z_2 = R_2 + \frac{1}{C_2 s}$$

Hence

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2 + \frac{1}{C_2 s}}{R_1 + \frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}}$$

$$= \frac{R_2 C_2 s + 1}{(R_1 C_2 + R_2 C_2) s + 1 + \frac{C_2}{C_1}}$$

If we change R_1 to b_1, R_2 to b_2, C_1 to $1/k_1$, C_2 to $1/k_2$, then we obtain

$$\frac{R_2 C_2 s + 1}{(R_1 + R_2) C_2 s + 1 + \frac{C_2}{C_1}} = \frac{b_2 \frac{1}{k_2} s + 1}{(b_1 + b_2)\frac{1}{k_2} s + 1 + \frac{k_1}{k_2}}$$

or

$$\frac{X_o(s)}{X_i(s)} = \frac{b_2 s + k_2}{(b_1 + b_2) s + k_2 + k_1} = \frac{b_2 s + k_2}{(b_1 s + k_1) + (b_2 s + k_2)}$$

The analogous mechanical system is shown below.