Absolute Value Definition

The absolute value of a number x, denoted by |x|, is defined by the formula

$$|x| = \begin{cases} x, & x \geq \\ -x, & x < 0 \end{cases}$$

Absolute Value Properties

1. $|-a| = |a|$ — A number and its additive inverse or negative have the same absolute value.

2. $|ab| = |a||b|$ — The absolute value of a product is the product of the absolute values.

3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ — The absolute value of a quotient is the quotient of the absolute values

4. $|a+b| \leq |a| + |b|$ — The triangle inequality. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Note that $|-a| \neq -|a|$ For example, $|-3| = 3$, whereas $-|3| = -3$. If a and b differ in sign, then $|a+b|$ is less than $|a| + |b|$. In all other cases, $|a+b|$ equals $|a| + |b|$ Absolute value bars in expressions like $|-3+5|$ work like parentheses: We do the arithmetic inside before taking the absolute value.

The following statements are all consequences of the definition of absolute value and are often helpful when solving equations or inequalities involving absolute values

1. $|x| = a$ if and only if $x = \pm a$
2. $|x| < a$ if and only if $-a < x < a$
3. $|x| > a$ if and only if $x > a$ or $x < -a$
4. $|x| \leq a$ if and only if $-a \leq x \leq a$
5. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

The symbol $\Leftrightarrow$ is often used by mathematicians to denote the "if and only if" logical relationship. It also means "implies and is implied by."

EXAMPLE3: Solve the inequality (عدم المساواة) and show the solution set on the real line, (a). $|2x - 3| \leq 1$ (b). $|2x - 3| \geq 1$.

Solution:

(a). $|2x - 3| \leq 1$

$-1 \leq 2x - 3 \leq 1$ Property 4

$2 \leq 2x \leq 4$ Add 3.

$1 \leq x \leq 2$ Divide by 2

The solution set is the closed interval [1, 2] (figure a).

(b). $|2x - 3| \geq 1$

$2x - 3 \geq 1$ or $2x - 3 \leq -1$ Property 5

$x - \frac{3}{2} \geq \frac{1}{2}$ or $x - \frac{3}{2} \leq -\frac{1}{2}$ Divide by 2

$x \geq 2$ or $x \leq 1$ Add $\frac{3}{2}$.

The solution set is (figure b).

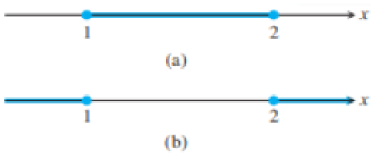


FIGURE : The solution sets (a) [1, 2] and (b) $(-\infty, 1] \cup [2, \infty)$ in Example 6.

EXAMPLE1: Solving an Equation with Absolute Values. Solve the equation $|2x - 3| = 7$.

Solution: By Property 1 above, $2x - 3 = \pm 7$, so there are two possibilities:

$2x - 3 = 7$ $\quad$ $2x - 3 = -7$

$2x = 10$ $\quad$ $2x = -4$

$x = 5$ $\quad$ $x = -2$

The solution of $|2x - 3| = 7$ $\quad$ $x = 5$ and $x = -2$

EXAMPLE2: Solving an Inequality Involving Absolute Values. Solve the inequality $\left|5-\frac{2}{x}\right|<1$.

Solution: We have

$$\left|5-\frac{2}{x}\right|<1$$
$$\Leftrightarrow -1<5-\frac{2}{x}<1 \text{ property 2}$$
$$\Leftrightarrow -6<-\frac{2}{x}<-4 \text{ Subtract 1}$$
$$\Leftrightarrow 3>\frac{1}{x}>2 \quad \text{Multiply by} \frac{1}{2}$$
$$\Leftrightarrow \frac{1}{3}<x<\frac{1}{2} \quad \text{Take reciprocals}$$

Notice how the various rules for inequalities were used here. Multiplying by a negative number reverses the inequality. So does taking reciprocals in an inequality in which both sides are positive. The original inequality holds if and only if $(\frac{1}{3}<x<\frac{1}{2})$. The solution set is the open interval (1/3, 1/2).

EXAMPLE3: Solve the inequality (عدم المساواة) and show the solution set on the real line, (a). $|2x-3|\leq 1$ (b). $|2x-3|\geq 1$.

Solution:

(a). $|2x-3|\leq 1$

$-1\leq 2x-3\leq 1$ Property 4

$2\leq 2x\leq 4$ Add 3.

$1\leq x\leq 2$ Divide by 2

The solution set is the closed interval [1, 2] (figure a).

(b). $|2x-3|\geq 1$

$2x-3\geq 1$ or $2x-3\leq -1$ Property 5

$x-\frac{3}{2}\geq\frac{1}{2}$ or $x-\frac{3}{2}\leq-\frac{1}{2}$ Divide by 2

$x\geq 2$ or $x\leq 1$ Add $\frac{3}{2}$.

The solution set is (figure b).

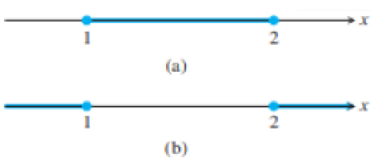


FIGURE : The solution sets (a) [1, 2] and (b) $(-\infty, 1]\cup[2, \infty)$ in Example 6.

Steps for Solving Linear Absolute Value Equations:

1. Isolate the absolute value.

2. Identify what the isolated absolute value is set equal to…
   a. If the absolute value is set **equal to zero**, remove absolute value symbols & solve the equation to get **one solution**.

   b. If the absolute value is set **equal to a negative** number, there is **no solution.**

   c. If the absolute value is set **equal to a positive** number, set the argument (*expression within the absolute value*) equal to the number **and** set it equal to the opposite of the number, using an 'or' statement in between the two equations. Then solve each equation separately to get **two solutions**.

Examples:

a. $|3x + 12| + 7 = 7$

$|3x + 12| = 0$

Because this equals **0**, there is **ONE** solution.

$3x + 12 = 0$

$3x = -12$

$\boldsymbol{x = -4}$

b. $|3x - 7| + 7 = 2$

$|3x - 7| = -5$

Because this equals a **negative** number, there is **NO** solution.

No Solution

c. $|3x - 7| + 7 = 9$

$|3x - 7| = 2$

Because this equals a **positive** number there are **TWO** sltns.

$3x - 7 = 2$ or $3x - 7 = -2$

$3x = 9$ or $3x = 5$

$\boldsymbol{x = 3}$ or $\boldsymbol{x = \frac{5}{3}}$

d. $|x + 5| = |2x - 1|$ → **Set up two Equations**

$x + 5 = +(2x - 1)$ or $x + 5 = -(2x - 1)$

$\boldsymbol{x = 6}$ or $x + 5 = -2x + 1 \rightarrow 3x = -4 \rightarrow \boldsymbol{x = -\frac{4}{3}}$

d. $|3x + 4| + 5 \leq 3$

$|3x + 4| \leq -2$

No Solution

e. $2|x - 1| - 4 \geq 2$

$2|x - 1| \geq 6$

$|x - 1| \geq 3$

$x - 1 \geq 3 \quad OR \quad x - 1 \leq -3$

$x \geq 4 \quad OR \quad x \leq -2$

$(-\infty, -\mathbf{2}] \cup [\mathbf{4}, \infty)$

f. $|x - 6| + 6 \geq -4$

$|x - 6| \geq -10$

All Real Numbers

Logarithmic Functions

General Formula

$y = \log_b x$ is equivalent to $b^y = x$

y is called the **logarithm**, b is called the **base**, and x is called the **argument**.

Example 1 Converting from Logarithmic Form to Exponential Form

Rewrite the logarithmic equations in exponential form.

a. $\log_2 32 = 5$ **b.** $\log_{10}\left(\frac{1}{1000}\right) = -3$ **c.** $\log_5 1 = 0$

Solution:

Logarithmic Form		Exponential Form
a. $\log_2 32 = 5$	$\Leftrightarrow$	$2^5 = 32$
b. $\log_{10}\left(\frac{1}{1000}\right) = -3$	$\Leftrightarrow$	$10^{-3} = \frac{1}{1000}$
c. $\log_5 1 = 0$	$\Leftrightarrow$	$5^0 = 1$

2. Evaluating Logarithmic Expressions

Example 2 Evaluating Logarithmic Expressions

Evaluate the logarithmic expressions.

a. $\log_{10} 10{,}000$ **b.** $\log_5\left(\frac{1}{125}\right)$ **c.** $\log_{1/2}\left(\frac{1}{8}\right)$

d. $\log_b b$ **e.** $\log_c\left(c^7\right)$ **f.** $\log_3\left(\sqrt[4]{3}\right)$

Solution:

a. $\log_{10} 10{,}000$ is the exponent to which 10 must be raised to obtain 10,000.

$y = \log_{10} 10{,}000$ Let y represent the value of the logarithm.

$10^y = 10{,}000$ Rewrite the expression in exponential form.

$y = 4$

Therefore, $\log_{10} 10{,}000 = 4$.

b. $\log_5\left(\frac{1}{125}\right)$ is the exponent to which 5 must be raised to obtain $\frac{1}{125}$.

$y = \log_5\left(\frac{1}{125}\right)$ Let y represent the value of the logarithm.

$5^y = \frac{1}{125}$ Rewrite the expression in exponential form.

$y = -3$

Therefore, $\log_5\left(\frac{1}{125}\right) = -3$.

c. $\log_{1/2}\left(\frac{1}{8}\right)$ is the exponent to which $\frac{1}{2}$ must be raised to obtain $\frac{1}{8}$.

$y = \log_{1/2}\left(\frac{1}{8}\right)$ Let y represent the value of the logarithm.

$\left(\frac{1}{2}\right)^y = \frac{1}{8}$ Rewrite the expression in exponential form.

$y = 3$

Therefore, $\log_{1/2}\left(\frac{1}{8}\right) = 3$.

d. $\log_b b$ is the exponent to which b must be raised to obtain b.

$y = \log_b b$ Let y represent the value of the logarithm.

$b^y = b$ Rewrite the expression in exponential form.

$y = 1$

Therefore, $\log_b b = 1$.

e. $\log_c\left(c^7\right)$ is the exponent to which c must be raised to obtain c^7.

$y = \log_c\left(c^7\right)$ Let y represent the value of the logarithm.

$c^y = c^7$ Rewrite the expression in exponential form.

$y = 7$

Therefore, $\log_c\left(c^7\right) = 7$.

f. $\log_3\left(\sqrt[4]{3}\right) = \log_3\left(3^{1/4}\right)$ is the exponent to which 3 must be raised to obtain $3^{1/4}$.

$y = \log_3(3^{1/4})$ Let y represent the value of the logarithm.

$3^y = 3^{1/4}$ Rewrite the expression in exponential form.

$$y = \frac{1}{4}$$

Therefore, $\log_3\left(\sqrt[4]{3}\right) = \frac{1}{4}$

3. The Common Logarithmic Function

Example 3 Evaluating Common Logarithms on a Calculator

Evaluate the common logarithms. Round the answers to four decimal places.

a. $\log 420$ **b.** $\log\left(8.2 \times 10^9\right)$ **c.** $\log(0.0002)$

Solution:

a. $\log 420 \approx 2.6232$

b. $\log\left(8.2 \times 10^9\right) \approx 9.9138$

c. $\log(0.0002) \approx -3.6990$

4. Graphs of Logarithmic Functions

Example 4 Graphing Logarithmic Functions

Graph the functions and compare the graphs to examine the effect of the base on the shape of the graph.

a. $y = \log_2 x$ **b.** $y = \log x$

Solution:

We can write each equation in its equivalent exponential form and create a table of values (Table 10-5). To simplify the calculations, choose integer values of y and then solve for x.

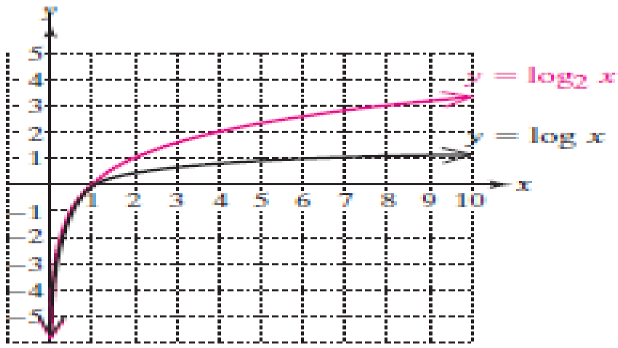
$$y = \log_2 x \quad \text{or} \quad 2^y = x \qquad y = \log x \quad \text{or} \quad 10^y = x$$

Choose values for y.

Table 10-5

$x = 2^y$	$x = 10^y$	y
$\frac{1}{8}$	$\frac{1}{1000}$	-3
$\frac{1}{4}$	$\frac{1}{100}$	-2
$\frac{1}{2}$	$\frac{1}{10}$	-1
1	1	0
2	10	1
4	100	2
8	1000	3

Solve for x.



Example 5 Graphing a Logarithmic Function

Graph $y = \log_{1/4} x$.

Solution:

The equation $y = \log_{1/4} x$ can be written in exponential form as $(\frac{1}{4})^y = x$. By choosing several values for y, we can solve for x and plot the corresponding points (Table 10-6).

The expression $y = \log_{1/4} x$ defines a decreasing logarithmic function (Figure 10-15). Notice that the vertical asymptote, domain, and range are the same for both increasing and decreasing logarithmic functions.

Table 10-6

$x = (\frac{1}{4})^y$	y
64	-3
16	-2
4	-1
1	0
$\frac{1}{4}$	1
$\frac{1}{16}$	2
$\frac{1}{64}$	3

↑ Solve for x.

↑ Choose values for y.

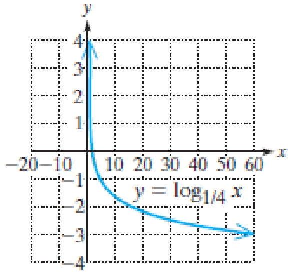


Figure 10-15

Example 1 Graphing $f(x) = e^x$

Graph the function defined by $f(x) = e^x$.

Solution:

Because the base of the function is greater than 1 ($e \approx 2.718281828$), the graph is an increasing exponential function. We can use a calculator to evaluate $f(x) = e^x$ at several values of x.

If you are using your calculator correctly, your answers should match those found in Table 10-8. Values are rounded to 3 decimal places. The corresponding graph of $f(x) = e^x$ is shown in Figure 10-16.

Table 10-8

x	$f(x) = e^x$
−3	0.050
−2	0.135
−1	0.368
0	1.000
1	2.718
2	7.389
3	20.086

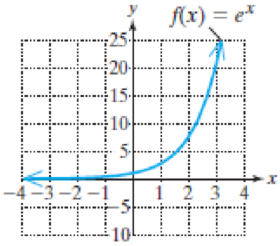


Figure 10-16

Properties of Logarithms

1. Properties of Logarithms

You have already been exposed to certain properties of logarithms that follow directly from the definition. Recall

$y = \log_b x$ is equivalent to $b^y = x$ for $x > 0$, $b > 0$, and $b \neq 1$

The following properties follow directly from the definition.

$\log_b 1 = 0$ Property 1

$\log_b b = 1$ Property 2

$\log_b b^p = p$ Property 3

$b^{\log_b x} = x$ Property 4

Example 1 Applying the Properties of Logarithms to Simplify Expressions

Use the properties of logarithms to simplify the expressions. Assume that all variable expressions within the logarithms represent positive real numbers.

a. $\log_8 8 + \log_8 1$ **b.** $10^{\log(x+2)}$ **c.** $\log_{1/2}\left(\frac{1}{2}\right)^x$

Solution:

a. $\log_8 8 + \log_8 1 = 1 + 0 = 1$ Properties 2 and 1

b. $10^{\log(x+2)} = x + 2$ Property 4

c. $\log_{1/2}\left(\frac{1}{2}\right)^x = x$ Property 3

Home Work

Skill Practice Rewrite the logarithmic equations in exponential form.

1. $\log_3 9 = 2$ **2.** $\log_{10}\left(\frac{1}{100}\right) = -2$ **3.** $\log_8 1 = 0$

Skill Practice Evaluate the logarithmic expressions.

4. $\log_{10} 1000$ **5.** $\log_4\left(\frac{1}{16}\right)$ **6.** $\log_{1/3} 3$

7. $\log_x x$ **8.** $\log_b (b^{10})$ **9.** $\log_5 (\sqrt[3]{5})$

Skill Practice Use the properties of logarithms to simplify the expressions.

1. $\log_5 1 + \log_5 5$ **2.** $15^{\log_{15} 7}$ **3.** $\log_{1/3}\left(\frac{1}{3}\right)^c$