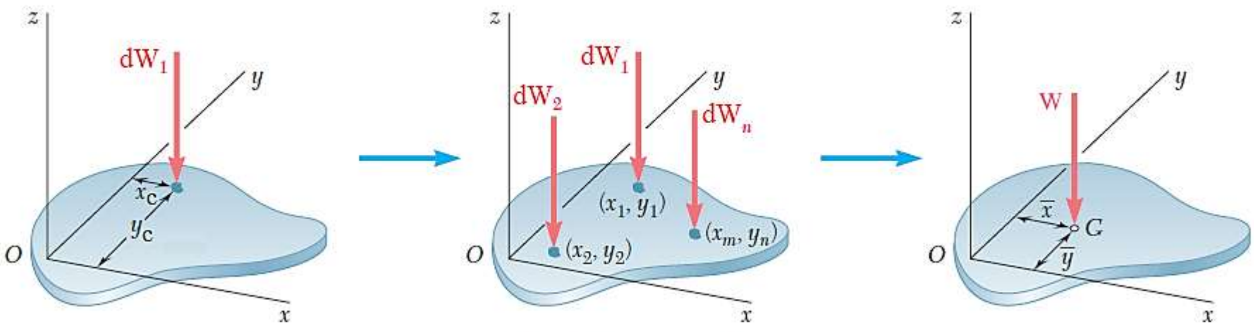
Chapter six: Centroid and Centers of Gravity

6.1 Introduction

A body is composed of an infinite number of particles of differential size, then each of these particles will have a weight d_W. These weights will form an approximately parallel force system, and the resultant of this system is the total weight of the body, which passes through a single point G called the *center of gravity*.



(a) Single element of the plate (b) Multiple elements of the plate (c) Center of gravity

6.1.1 Centers of Gravity of the Body

To determine mathematically the location of the center of gravity of any body, we apply the principle of moments to the parallel system of weight forces. The resultant of the weight forces acting on all elements given by:

$$W = \int dW$$

The location of the center of gravity, measured from the *y*-axis, is determined by equating the moment of *W* about the *y*-axis, the moment about this axis of the elemental weight is:

$$dMy = x_c \cdot dW$$

The sum of the moments of the weights of the particles about this same axis is:

$$My = \int x_c \cdot dW$$

This sum of moments must equal $\bar{x} \cdot W$. Thus,

$$\bar{x} \cdot W = My \quad \Longrightarrow \quad \bar{x} \cdot \int dW = \int x_c \cdot dW$$

Similarly, if moments are summed about the *x*-axis,

$$\bar{y} \cdot W = Mx \quad \Longrightarrow \quad \bar{y} \cdot \int dW = \int y_c \cdot dW$$

Therefore, the location of the center of gravity *G* with respect to the *x*, *y* axes becomes

$$\bar{x} = \frac{My}{W} = \frac{\int x_c \cdot dW}{\int dW}, \qquad \bar{y} = \frac{Mx}{W} = \frac{\int y_c \cdot dW}{\int dW}$$

where:

$\bar{x}, \bar{y}$: the coordinates of the center of gravity *G*.

x_c, y_c: the coordinates of each particle in the body.

6.1.2 Centers of Mass of the Body

In order to study the dynamic response or accelerated motion of a body, it becomes important to locate the body's center of mass *Cm*. As weight can be expressed as a product of mass and acceleration due to gravity $dW = g\,dm$, we can write the Eq. as:

$$\bar{x} = \frac{\int x_c \cdot g\,dm}{\int g\,dm}, \qquad \bar{y} = \frac{\int y_c \cdot g\,dm}{\int g\,dm}$$

Since g is constant, it cancels out, and so

$$\bar{x} = \frac{My}{m} = \frac{\int x_c \cdot dm}{\int dm}, \qquad \bar{y} = \frac{Mx}{m} = \frac{\int y_c \cdot dm}{\int dm}$$

6.1.3 Centroid of a Volume

If the body is made from a homogeneous material, then its density ρ will be constant. Therefore, a differential element of volume dV has a mass $dm = \rho\,dV$. we can write the Eq. as:

$$\bar{x} = \frac{\int x_c \cdot \rho \, dV}{\int \rho \, dV}, \qquad \bar{y} = \frac{\int y_c \cdot \rho \, dV}{\int \rho \, dV}$$

Since ρ is constant, it cancels out, and so

$$\bar{x} = \frac{My}{V} = \frac{\int x_c \cdot dV}{\int dV}, \qquad \bar{y} = \frac{Mx}{V} = \frac{\int y_c \cdot dV}{\int dV}$$

6.1.4 Centroid of an Area

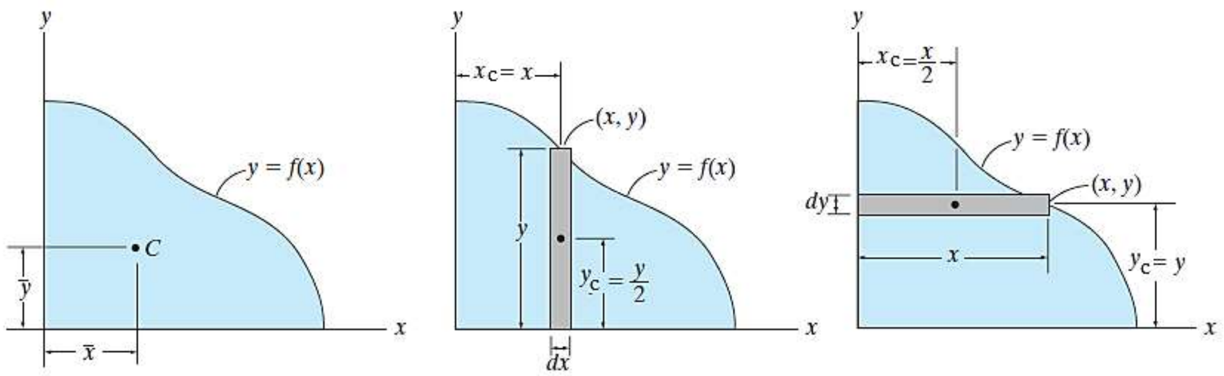
Further, the volume can be expressed as a product of cross-sectional area and thickness, $dV = t\, dA$ and hence we can write

$$\bar{x} = \frac{\int x_c \cdot t \, dA}{\int t \, dA}, \qquad \bar{y} = \frac{\int y_c \cdot t \, dA}{\int t \, dA}$$

Again, if the thickness or depth of the body is constant, then we can take it outside the integral sign. Therefore,

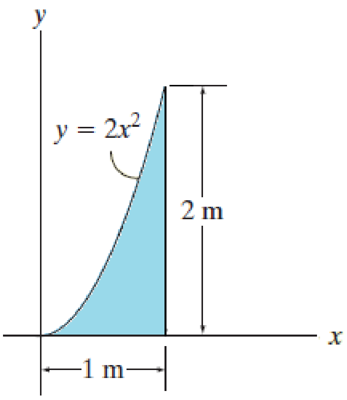
$$\bar{x} = \frac{My}{A} = \frac{\int x_c \cdot dA}{\int dA}, \qquad \bar{y} = \frac{Mx}{A} = \frac{\int y_c \cdot dA}{\int dA}$$

These integrals can be evaluated by performing a *single integration* if we use a *rectangular strip* for the differential area element. For example, if a vertical strip is used, Fig. b , the area of the element is $dA = y\, dx$, and its centroid is located at $x_c = x$ and $y_c = y/2$. If we consider a horizontal strip, Fig. c , then $dA = x\, dy$, and its centroid is located at $x_c = x/2$ and $y_c = y$.



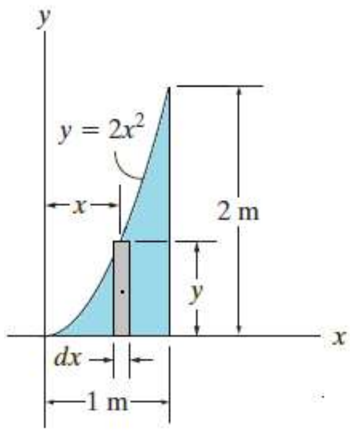
(a) (b) (c)

Example No. 1: Locate the centroid of the area shown in Figure.



Solution:

Method I: by vertical strip



$$\bar{x} = \frac{My}{A} = \frac{\int_A x_c \cdot dA}{\int_A dA}, \qquad \bar{y} = \frac{Mx}{A} = \frac{\int_A y_c \cdot dA}{\int_A dA}$$

$$x_c = x, \quad y_c = \frac{y}{2}$$

$$dA = y\,dx = 2x^2 dx$$

$$A = \int_A dA = \int_0^1 2x^2 dx = 2\frac{x^3}{3}\bigg]_0^1 = \frac{2}{3}\ m^2$$

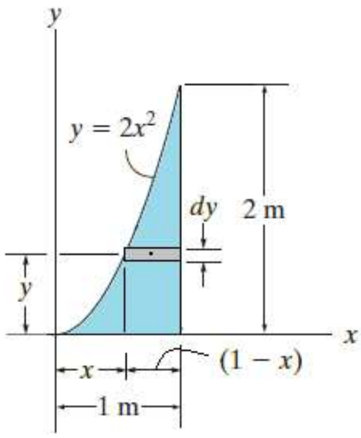
$$My = \int_A x_c \cdot dA = \int_0^1 x \cdot 2x^2 dx = \int_0^1 2x^3 dx = 2\frac{x^4}{4}\bigg]_0^1 = 0.5\ m^3$$

$$Mx = \int_A y_c \cdot dA = \int_0^1 \frac{y}{2} \cdot 2x^2 dx = \int_0^1 2x^2 \cdot x^2 dx = \int_0^1 2x^4 dx = 2\frac{x^5}{5}\bigg]_0^1 = \frac{2}{5}\ m^3$$

$$\bar{x} = \frac{My}{A} = \frac{0.5}{2/3} = 0.75\ m$$

$$\bar{y} = \frac{Mx}{A} = \frac{2/5}{2/3} = 0.6\ m$$

Method II: **by horizontal strip**



$$\bar{x} = \frac{My}{A} = \frac{\int_A x_c \cdot dA}{\int_A dA}, \qquad \bar{y} = \frac{Mx}{A} = \frac{\int_A y_c \cdot dA}{\int_A dA}$$

$$x_c = x + \frac{(1-x)}{2} = \frac{1}{2}(x+1)$$

$$y_c = y$$

$$dA = (1-x)\,dy$$

$$y = 2x^2 \quad \Longrightarrow \quad x = \sqrt{\frac{y}{2}}$$

$$A = \int_A dA = \int_0^2 (1-x)\,dy = \int_0^2 \left(1 - \sqrt{\frac{y}{2}}\right) dy$$

$$= \int_0^2 \left(1 - (y/2)^{1/2}\right) dy = \left[y - \frac{(y/2)^{3/2} \times 2}{3/2}\right]_0^2 = \frac{2}{3}\,m^2$$

$$My = \int_A x_c \cdot dA = \int_0^2 \frac{1}{2}(x+1)\cdot(1-x)\,dy = \int_0^2 \frac{1}{2}\left(\sqrt{\frac{y}{2}} + 1\right)\cdot\left(1 - \sqrt{\frac{y}{2}}\right) dy$$

$$= \frac{1}{2}\int_0^2 \left(1 - \frac{y}{2}\right) dy = \frac{1}{2}\left[y - \frac{y^2}{4}\right]_0^2 = 0.5\,m^3$$

$$Mx = \int_A y_c \cdot dA = \int_0^2 y\cdot(1-x)\,dy = \int_0^2 y\cdot\left(1 - \sqrt{\frac{y}{2}}\right) dy = \int_0^2 \left(y - \frac{y^{3/2}}{\sqrt{2}}\right) dy$$

$$= \left[\frac{y^2}{2} - \frac{y^{5/2}}{\sqrt{2}\times 5/2}\right]_0^2 = \frac{2}{5}\,m^3$$

$$\bar{x} = \frac{My}{A} = \frac{0.5}{2/3} = 0.75\,m$$

$$\bar{y} = \frac{Mx}{A} = \frac{2/5}{2/3} = 0.6\,m$$