The Limits

A limit is the value that a function or sequence "approaches" as the input or index approaches some value. We say that the limit of $f(x)$ is as x approaches a and write this as $\lim_{x \to a} f(x) = L$

Proposition

1. If a and c are constants then $\lim_{x \to a} c = c$
2. $\lim_{x \to a} x = a$

Let $f_1(x) = L_1$ and $f_2(x) = L_2$ then

3. $\lim_{x \to a}\left(f_1(x) + f_2(x)\right) = L_1 + L_2$
4. $\lim_{x \to a}\left(f_1(x) - f_2(x)\right) = L_1 - L_2$
5. $\lim_{x \to a}\left(f_1(x) . f_2(x)\right) = L_1 . L_2$
6. $\lim_{x \to a}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{L_1}{L_2}$; $L_2 \neq 0$
7. If $\lim_{x \to a} g(x) = L$ then $\lim_{x \to a} f\left(g(x)\right) = f(L)$

Example

1. $\lim_{x \to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3} = \lim_{x \to 3} \frac{\sqrt{x} - \sqrt{3}}{(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})} = \lim_{x \to 3} \frac{1}{(\sqrt{x} + \sqrt{3})} = \frac{1}{2\sqrt{3}}$

2. $\lim_{x \to \infty} \frac{5x^2 + 3}{7x^2 + 2x - 5}$

   Divide top and bottom by x^2, then we get

   $$\lim_{x \to \infty} \frac{5x^2 + 3}{7x^2 + 2x - 5} = \lim_{x \to \infty} \frac{5 + (3/x^2)}{7 + (2/x) - (5/x^2)} = \frac{5}{7}$$

Example: Find the following limits

1. $\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}=\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}=\lim_{x\to 3}\frac{1}{(\sqrt{x}+\sqrt{3})}=\frac{1}{2\sqrt{3}}$

2. $\lim_{x\to\infty}\frac{5x^2+3}{7x^2+2x-5}$

   Divide top and bottom by x^2, then we get

   $$\lim_{x\to\infty}\frac{5x^2+3}{7x^2+2x-5}=\lim_{x\to\infty}\frac{5+(3/x^2)}{7+(2/x)-(5/x^2)}=\frac{5}{7}$$

3.

$$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}=\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\times\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

$$=\lim_{x\to 0}\frac{4+x-4}{x\sqrt{4+x}+2}=\lim_{x\to 0}\frac{x}{x\sqrt{4+x}+2}=\frac{1}{4}$$

Derivatives

Rules for finding derivatives

1. **Constant Rule** *The derivative of a constant is always zero. That is if* $f(x) = c$ *then* $f\grave{}(x) = 0.$
2. **Power Rule** *The derivative of a power function,* $\boldsymbol{f(x) = x^n}$ *. Here* n *is a number of any kind: integer, rational, positive, negative, even irrational, as in* x*. The derivative is* $\boldsymbol{f\grave{}(x) = nx^{n-1}}$
3. *Multiplication by constant: The derivative of* $cf(x)$ *is* $cf\grave{}(x)$
4. *Sum Rule: The derivative of* $f(x) + g(x)$ *is* $f\ (x) + g\ (x)$
5. *Difference Rule: The derivative of* $f(x) - g(x)$ *is* $f\grave{}(x) - g\grave{}(x)$
6. *Product Rule: The derivative of the product of two functions is not the product of the functions' derivatives; rather, it is described by the equation below*

$$\frac{d}{dx}(f(x)\times g(x)) = f(x)\times g\grave{}(x) + f\grave{}(x)\times g(x)$$

7 . Quotient Rule: The derivative of the quotient of two functions is not the quotient of the functions' derivatives; rather, it is described by the equation below:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\times f\grave{}(x)-f(x)\times g\grave{}(x)}{(g(x))^2}$$

Example: *Find derivatives of the functions*

1. $y=(2x+3)^4 \Rightarrow \frac{dy}{dx}=4(2x+3)^3\times 2=8(2x+3)^3$

2. $y=\sqrt{x^2+3x} \Rightarrow y=(x^2+3x)^{1/2}$

$$\frac{dy}{dx}=\frac{1}{2}(x^2+3x)^{-1/2}(2x+3)=\frac{(2x+3)}{2\sqrt{x^2+3x}}$$

3. $y=\frac{x}{\sqrt{x^2+1}} \Rightarrow \frac{dy}{dx}=\frac{\sqrt{x^2+1}\times 1-x\times(1/2)(x^2+1)^{-1/2}\times 2x}{x^2+1}$

$$=\frac{\sqrt{x^2+1}-\frac{x^2}{\sqrt{x^2+1}}}{x^2+1}=\frac{\frac{x^2+1-x^2}{\sqrt{x^2+1}}}{\sqrt{x^2+1}\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}}$$

Trigonometric functions

Definitions of trigonometric functions for a right triangle A right triangle is a triangle with a right angle (90°) For every angle θ in the triangle, there is the side of the triangle adjacent to it, the side opposite of it and the hypotenuse such that $a^2+b^2=c^2$.

Trigonometric functions of negative angles

$$\sin(-\theta)=-\sin\theta \quad , \quad \cos(-\theta)=\cos\theta \quad \text{and} \quad \tan(-\theta)=-\tan\theta$$

Some useful relationships among trigonometric functions

1. $\sin^2x+\cos^2x=1$, $\sec^2x-\tan^2x=1$, $\csc^2x-\cot^2x=1$
2. $\sin 2x=2\sin x\cos x$, $\cos 2x=\cos^2x-\sin^2x=1-2\sin^2x=2\cos^2x-1$
3. $\sin^2x=\frac{1-\cos 2x}{2}$, $\cos^2x=\frac{1+\cos 2x}{2}$

Derivatives of trigonometric functions

If u is a function x, the chain rule version of this differentiation rule is

$$1. \frac{d}{dx}(\sin u) = \cos u \,.\frac{du}{dx}$$

$$2. \frac{d}{dx}(\cos u) = -\sin u \,.\frac{du}{dx}$$

$$3. \frac{d}{dx}(\tan u) = \sec^2 u.\frac{du}{dx}$$

$$4. \frac{d}{dx}(\cot u) = -\csc^2 u.\frac{du}{dx}$$

$$5. \frac{d}{dx}(\sec u) = \sec u \tan u \,.\frac{du}{dx}$$

$$6. \frac{d}{dx}(\csc u) = -\csc u \cot u \,.\frac{du}{dx}$$

Example: Find derivatives of the functions

1. $y = \sin^2 x \quad \Rightarrow \quad y = (\sin x)^2 \quad \Rightarrow \quad \frac{dy}{dx} = 2\sin x \cos x = \sin 2x$
2. $y = \cos(x^2) \quad \Rightarrow \quad \frac{dy}{dx} = -2x\sin(x^2)$
3. $y = \tan\sqrt{x} \quad \Rightarrow \quad \frac{dy}{dx} = \sec^2\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$
4. $y = x^2 \sec 3x \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2 \sec 3x \tan 3x + 2x \sec 3x = x \sec 3x\,(2 + 3x\tan 3x)$
5. $y = \sqrt{\sin 2x} \quad \Rightarrow \quad y = (\sin 2x)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(\sin 2x)^{-1/2} \times \cos 2x \times 2$

$$= \frac{\cos 2x}{\sqrt{\sin 2x}}$$

Integral

If the function $F(x)$ is an ant derivative of $f(x)$ then the expression $F(x) + C$ where C is an arbitrary constant, is called the indefinite integral of $f(x)$ and we write

$$\int f(x)\,dx = F(x) + c$$

The integral of the power function

If u is a function of x, then

1. $\int u^n du = \frac{u^{n+1}}{n+1} + c \;; n \in \mathbb{R}\,,\; n \neq -1$

2. $\int \frac{du}{u} = \ln|u| + c$

Examples

1. $\int \sqrt{3x+1}\,dx = \frac{1}{3}\int 3\,(3x+1)^{\frac{1}{2}}dx = \frac{1}{3}(3x+1)^{\frac{3}{2}} \times \frac{2}{3} + c = \frac{2}{9}(3x+1)^{\frac{3}{2}} + c$

2. $\int \frac{xdx}{\sqrt{x^2-3}} = \frac{1}{2}\int 2x(x^2-3)^{-1/2}dx = \frac{1}{2}(x^2-3)^{1/2} \times 2 + c = \sqrt{x^2-3} + c$

3. $\int \frac{xdx}{x^2-3} = \frac{1}{2}\ln|x^2-3| + c$

The integrals of trigonometric functions

If u is a function of x, then

1. $\int \sin u\, du = -\cos u + c$
2. $\int \cos u\, du = \sin u + c$
3. $\int \sec^2 u\, du = \tan u + c$
4. $\int \csc^2 u\, du = -\cot u + c$
5. $\int \sec u \tan u\, du = \sec u + c$
6. $\int \csc u \cot u\, du = -\csc u + c$
7. $\int \tan u\, du = -\ln|\cos u| + c = \ln|\sec u| + c$
8. $\int \cot u\, du = \ln|\sin u| + c = -\ln|\csc u| + c$

Example

4. $\int \sqrt{1+\cos 2x}\, dx = \int \sqrt{2\cos^2 x}\, dx = \int \sqrt{2}\, \cos x\, dx = \sqrt{2}\sin x + c$
5. $\int \frac{\sin 2x}{\sqrt{1+\cos 2x}} dx = \int (1+\cos 2x)^{-1/2} \sin 2x\, dx = -\sqrt{1+\cos 2x} + c$
6. $\int \frac{\sin 2x}{1+\cos 2x} dx = \frac{1}{2}\int \frac{2\sin 2x}{1+\cos 2x} dx = \frac{1}{2}\ln(1+\cos 2x) + c$