# Trigonometric Functions Inverse الدوال المثلثية المعكوسة

**<u>بعض الخصائص :</u>**

$1.\ \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$ $\qquad$ $2.\ \cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

$3.\ \cos^{-1}(-x) = \pi - \cos^{-1}x$ $\qquad$ $4.\ \tan^{-1}(-x) = -\tan^{-1}x$

$5.\ \sec^{-1}(-x) = \pi - \sec^{-1}x$ $\qquad$ $6.\ \sec^{-1}x = \cos^{-1}\frac{1}{x}$

$7.\ \csc^{-1}x = \sin^{-1}\frac{1}{x}$

**<u>مشتقات الدوال المثلثية العكسية :</u>**

$$1.\ \frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}}.\frac{du}{dx} \qquad 2.\ \frac{d}{dx}(\cos^{-1}u) = \frac{-1}{\sqrt{1-u^2}}.\frac{du}{dx}$$

$$3.\ \frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2}.\frac{du}{dx} \qquad 4.\ \frac{d}{dx}(\cot^{-1}u) = \frac{-1}{1+u^2}.\frac{du}{dx}$$

$$5.\ \frac{d}{dx}(\sec^{-1}u) = \frac{1}{|u|\sqrt{u^2-1}}.\frac{du}{dx} \qquad 6.\ \frac{d}{dx}(\csc^{-1}u) = \frac{-1}{|u|\sqrt{u^2-1}}.\frac{du}{dx}$$

<u>Example:-</u>

مثال (٧) جد $y'$ للدوال التالية :

$1.\ y = \tan^{-1}\frac{x-1}{x+1}$ $\qquad$ $2.\ y = x\ \sin^{-1}x + \sqrt{1-x^2}$

$3.\ y = \sqrt{x^2-1} - \sec^{-1}x$ $\qquad$ $4.\ y = x\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2}$

الحل :

$$1.\ y' = \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2} \times \frac{x+1-(x-1)}{(x+1)^2} = \frac{1}{\frac{(x+1)^2+(x-1)^2}{(x+1)^2}} \times \frac{x+1-x+1}{(x+1)^2}$$

$$= \frac{2}{x^2+2x+1+x^2-2x+1} = \frac{1}{x^2+1}$$

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$$2.\ y' = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x + \frac{-2x}{2\sqrt{1-x^2}} = \sin^{-1}x$$

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$$3.\ y' = \frac{2x}{2\sqrt{x^2-1}} - \frac{1}{x\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}} - \frac{1}{x\sqrt{x^2-1}} = \frac{x^2-1}{x\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x}$$

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$$4.\ y' = \frac{2x}{\sqrt{1-4x^2}} + \cos^{-1}2x - \frac{-8x}{4\sqrt{1-4x^2}} = \cos^{-1}2x$$

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# Hyperbolic Functions

1) $\sinh x = \frac{e^x - e^{-x}}{2}$

2) $\cosh x = \frac{e^x + e^{-x}}{2}$

3) $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4) $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

5) $\cosh x + \sinh x = e^x$

6) $\cosh x - \sinh x = e^{-x}$

7) $\tanh x = \frac{\sinh x}{\cosh x}$

8) $\cosh x = \frac{\cosh x}{\sinh x}$

9) $\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

10) $\text{csch}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

11) $\cosh^2 x - \sinh^2 = 1$

12) $\sinh^2 x = \cosh^2 x - 1$

13) $\cosh^2 x = 1 + \sinh^2$

14) $1 - \tanh^2 x = \text{sech}^2 x$

15) $\coth^2 x - 1 = \text{csch}^2 x$

16) $\sinh(x \mp y) = \sinh x \cosh y \mp \sinh y \cosh x$

17) $\cosh(x \mp y) = \cosh x \cosh y \mp \sinh x \sinh y$

18) $\sinh 2x = 2\sinh x \cosh y$

19) $\cosh 2x = \cosh^2 x + \sinh^2 x$

20) $\sinh^2 x = \frac{\cosh 2x - 1}{2}$

21) $\cosh^2 x = \frac{\cosh 2x + 1}{2}$

س١/ اثبت ان $\cosh x + \sinh x = e^x$

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x$$

س٢/ اثبت ان $\cosh x - \sinh x = e^{-x}$

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = \frac{2e^{-x}}{2} = e^{-x}$$

## Prove that

$$\cosh^2 x - \sinh^2 x = 1\,.$$

To prove this, we start by substituting the definitions for $\sinh x$ and $\cosh x$:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 .$$

If we expand the two squares in the numerators, we obtain

$$\begin{aligned}(e^x + e^{-x})^2 &= e^{2x} + 2(e^x)(e^{-x}) + e^{-2x} \\ &= e^{2x} + 2 + e^{-2x}\end{aligned}$$

and

$$\begin{aligned}(e^x - e^{-x})^2 &= e^{2x} - 2(e^x)(e^{-x}) + e^{-2x} \\ &= e^{2x} - 2 + e^{-2x}\,,\end{aligned}$$

where in each case we use the fact that $(e^x)(e^{-x}) = e^{x+(-x)} = e^0 = 1$. Using these expansions in our formula, we obtain

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}\,.$$

Now we can move the factor of $\frac{1}{4}$ out to the front, so that

$$\cosh^2 x - \sinh^2 x = \tfrac{1}{4}\left((e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right) .$$

If, finally, we remove the inner brackets and simplify, we obtain

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) \\ &= \tfrac{1}{4} \times 4 \\ &= 1\,,\end{aligned}$$

| Derivatives of Hyperbolic Functions |
|---|
| $\frac{d}{dx}\sinh u = \cosh u\frac{du}{dx}$ |
| $\frac{d}{dx}\cosh u = \sinh u\frac{du}{dx}$ |
| $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u\frac{du}{dx}$ |
| $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u\frac{du}{dx}$ |
| $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u\tanh u\frac{du}{dx}$ |
| $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u\coth u\frac{du}{dx}$ |

Find $\frac{d}{dt}(\tanh\sqrt{1+t^2})$.

*Solution.* To do this we shall have to use the chain rule. Let $u=\sqrt{1+t^2}$. Then

$$\begin{aligned}
\frac{d}{dt}(\tanh\sqrt{1+t^2}) &= \frac{d}{dt}(\tanh u)\\
&= \left(\frac{d}{du}(\tanh u)\right)\frac{du}{dt}\\
&= (\operatorname{sech}^2 u)\,\tfrac{1}{2}(1+t^2)^{-1/2}\,(2t)\\
&= \frac{t\operatorname{sech}^2\sqrt{1+t^2}}{\sqrt{1+t^2}}
\end{aligned}$$

Find the derivative.

$$y = 6x^{-4} - \cosh^{-1}(4x^7)$$

Applying chain rule, we get

$$y\prime = -24x^{-5} - \left[\frac{1}{\sqrt{(4x^7)^2-1}}\right](28x^6)$$

$$y\prime = -\frac{24}{x^5} - \frac{28x^6}{\sqrt{16x^{14}-1}}$$

Show that $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$

*Solution* We know that $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

and using the quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$

we have $\frac{d}{dx}\tanh(x) = \frac{1}{\cosh(x)} \times \cosh(x) - \frac{\sinh(x)}{\cosh^2(x)} \times \sinh(x)$

so $\frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x)$ ❑

| Inverse Hyperbolic Identities |
|---|
| $\operatorname{sech}^{-1}x = \cosh^{-1}\left(\frac{1}{x}\right)$ |
| $\operatorname{csch}^{-1}x = \sinh^{-1}\left(\frac{1}{x}\right)$ |
| $\coth^{-1}x = \tanh^{-1}\left(\frac{1}{x}\right)$ |

| Derivatives of Inverse Hyperbolic Functions | |
|---|---|
| $\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$ | |
| $\frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$ | $(u>1)$ |
| $\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}$ | $(\lvert u\rvert<1)$ |
| $\frac{d}{dx}\operatorname{csch}^{-1}u = \frac{-1}{\lvert u\rvert\sqrt{1+u^2}}\frac{du}{dx}$ | $(u\neq 0)$ |
| $\frac{d}{dx}\operatorname{sech}^{-1}u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$ | $(0<u<1)$ |
| $\frac{d}{dx}\coth^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}$ | $(\lvert u\rvert>1)$ |

Prove that $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$.

**SOLUTION 1** Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to $x$, we get

$$\cosh y\,\frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geqslant 0$, we have $\cosh y = \sqrt{1+\sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

Find $\frac{d}{dx}[\tanh^{-1}(\sin x)]$.

**SOLUTION** Using Table 6 and the Chain Rule, we have

$$\frac{d}{dx}[\tanh^{-1}(\sin x)] = \frac{1}{1-(\sin x)^2}\frac{d}{dx}(\sin x)$$

$$= \frac{1}{1-\sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x$$